Galois 2023 Day 8

I. Galois Theory for Finite Étale Covers

Prop $\phi: X \to S$ finite étale cover; $s: S \to X$ a section of ϕ (morphism s.t. $\phi \circ s = id_S$). Then s induces an isomorphism of S with an open and closed subscheme of X. If S connected, s maps S isomorphically onto a whole connected component of X.

To prove this, need lemma:

Lem $\phi: X \to S$, $\psi: Y \to X$ morphisms of schemes.

1. If $\phi \circ \psi$ is finite and ϕ separated, ψ finite.
2. If $\phi \circ \psi$ and ϕ finite étale, then so is ψ.

Proof of Prop / by lemma s is finite étale, hence its image is both open and closed in X. It's also injective, hence prop.

Cor. If $Z \to S$ connected S-scheme, $\phi_1, \phi_2 : Z \to X$ two S-morphisms to a finite étale S-scheme X with $\phi_1 \circ \bar{z} = \phi_2 \circ \bar{z}$ for some geometric point $\bar{z}: \mathrm{Spec}(\Omega) \to Z$, then $\phi_1 = \phi_2$.

- Given morphism of schemes, define $\mathrm{Aut}(X|S)$ to be the group of scheme automorphisms of X preserving ϕ. For a geometric point $\bar{s}: \mathrm{Spec}(\Omega) \to S$ there is a natural left action of $\mathrm{Aut}(X|S)$ on the geometric fibre $X_{\bar{s}} = X \times_S \mathrm{Spec}(\Omega)$ from base change.

Cor. If $\phi: X \to S$ is a connected finite étale cover, the nontrivial elements of $\mathrm{Aut}(X|S)$ act without fixed points on each geometric fibre. Hence $\mathrm{Aut}(X|S)$ is finite.

Construction Let $\phi: X \to S$ be an affine surjective morphism of schemes ; $G \subset \mathrm{Aut}(X|S)$ a finite subgroup. Define a ringed space $G\backslash X$ and a morphism $\pi: X \to G\backslash X$ of ringed spaces :

- as top. space, $G\backslash X$ is quotient of X by action of G
- π, natural projection
- Structure sheaf of $G\backslash X$, subsheaf $(\pi_* \mathcal{O}_X)^G$ of G-invariant elements in $\pi_* \mathcal{O}_X$.

Prop The ringed space $G\backslash X$ as constructed is a scheme ; morphism π is affine and surjective, and ϕ factors as $\phi = \psi \circ \pi$ with an affine morphism $\psi : G\backslash X \to S$.

Prop. $\phi: X \to S$ connected finite étale cover, and $G \subset \mathrm{Aut}(X|S)$ a finite group of S-automorphisms of X. Then $X \to G\backslash X$ is a finite étale cover of $G\backslash X$, and $G\backslash X$ is a finite étale cover of S.

- A connected finite étale cover $X \to S$ is Galois if its S-automorphism group acts transitively on geometric fibres.

Analogue of Galois Theorem (Proof same as topological one)

$\phi: X \to S$ finite étale Galois cover. If $Z \to S$ is a connected finite étale cover fitting into

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\phi"'] & Z \arrow[d, "\psi"] \\ & S \end{tikzcd}$$

then $\pi: X \to Z$ is a finite étale Galois cover, and actually $Z \cong H\backslash X$ with some subgroup H of $G = \mathrm{Aut}(X|S)$.

We get bijection between subgroups of G and intermediate covers Z ; the cover $\psi: Z \to S$ is Galois iff H normal, in which case $\mathrm{Aut}(Z|S) \cong G/H$.

Prop. (Serre) $\phi: X \to S$ connected finite étale cover. There is a morphism $\pi: P \to X$ such that $\phi \circ \pi : P \to S$ is a finite étale Galois cover ; every S-morphism from a Galois cover to X factors through P.

II. Group Schemes and Torsors

- S scheme. A group scheme over S is a morphism of schemes $p: G \to S$ that has a section $e: S \to G$ together with S-morphisms $\mathrm{mult}: G \times_S G \to G$, $\quad i: G \to G$, s.t.

$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "\mathrm{id} \times \mathrm{mult}"] \arrow[d, "\mathrm{mult} \times \mathrm{id}"'] & G \times_S G \arrow[d, "\mathrm{mult}"] \\ G \times_S G \arrow[r, "\mathrm{mult}"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\mathrm{id} \times e"] \arrow[d, "e \times \mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & G \times_S G \arrow[d, "\mathrm{mult}"] \\ G \times_S G \arrow[r, "\mathrm{mult}"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\mathrm{id} \times i"] \arrow[d, "i \times \mathrm{id}"'] \arrow[dr, "e \circ p"] & G \times_S G \arrow[d, "\mathrm{mult}"] \\ G \times_S G \arrow[r, "\mathrm{mult}"'] & G \end{tikzcd}$$

- G is <u>finite</u> if the structure morphism $p: G \to S$ is finite. Similarly, G is <u>finite flat</u> / <u>finite étale</u> if p is finite locally free / finite étale.
- ex. $S = \mathrm{Spec}(k)$. If $G = \mathrm{Spec}(A)$ with finite étale k-algebra A, $G \to S$ is a finite étale group scheme.

  Fibre of G over the geometric point given by an algebraic closure of k_s carries a group structure coming from that of G; compatible with the Galois action.
- S connected scheme, $G \to S$ a finite flat group scheme. A (left) <u>G-torsor</u> or <u>principal homogeneous space</u> over S is a finite locally free surjective morphism $X \to S$ together with a group action $\rho: G \times_S X \to X$ (defined by diagram) such that the map $(\rho, id): G \times_S X \to X \times_S X$ is an isomorphism.
- (lem) S connected scheme, $G \to S$ finite flat group scheme, $X \to S$ a scheme over S with left action $\rho: G \times_S X \to X$. These data define a G-torsor over S iff there exists a finite locally free surjective morphism s.t. $X \times_S Y \to Y$ is isomorphic as a Y-scheme with $G \times_S Y$-action, to $G \times_S Y$ acting on itself by left translations.
- (Prop.) S connected scheme, G finite étale group scheme over S.
  1. G is a constant group scheme Γ_S, then a G-torsor is the same as a finite étale Galois cover with group Γ.
  2. If $\exists$ morphism $S \to \mathrm{Spec}(k)$ with a field k, and G arises from an étale k-group scheme G_k by base change to S, then every G-torsor $Y \to S$ is a finite étale cover of S. Moreover, there is a finite separable extension $L|k$ such that $Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}\, L \to S \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$ is a Galois étale cover.

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Algebraic fundamental group

- Scheme S. denote Fet_S the category:
  - objects: finite étale covers of S
  - morphisms: morphisms of schemes over S.
- For objects $X \to S$, consider geometric fibre $X \times_S \mathrm{Spec}(\Omega)$ over geometric point $\bar{s}: \mathrm{Spec}(\Omega) \to S$; denote its underlying set $\mathrm{Fib}_{\bar{s}}(X)$.
- Given a morphism $X \to Y$ in Fet_S, induces morphism of schemes $X \times_S \mathrm{Spec}(\Omega) \to Y \times_S \mathrm{Spec}(\Omega)$, which induces set-theoretic map $\mathrm{Fib}_{\bar{s}}(X) \to \mathrm{Fib}_{\bar{s}}(Y)$.
- Call $\mathrm{Fib}_{\bar{s}}$ on Fet_S the <u>fibre functor</u> at the geometric point $\bar{s}$.
- Categorically, given functor $F: \mathcal{C}_1 \to \mathcal{C}_2$, an <u>automorphism</u> of F is a morphism of functors (natural trans) $F \to F$ with a two-sided inverse.

- $\mathrm{Aut}(F)$ group under composition of morphisms; automorphism group of F.

- Given scheme S and a geometric point $\bar{s} : \mathrm{spec}(\Omega) \to S$. define the <u>algebraic fundamental group</u> $\pi_1(S, \bar{s})$ the automorphism group of the fibre functor $\mathrm{Fib}_{\bar{s}}$ on Fet_S.