Galois 2023 Day 6

I. Intro to Schemes

- Setting: A ring, $\mathrm{Spec}(A)$ topological space whose points are prime ideals of A, basis of open sets given by the sets $D(f) := \{\mathfrak{p} : \mathfrak{p} \text{ is prime ideal with } f \notin \mathfrak{p}\}$ for each $f \in A$. Zariski Topology
- $\exists$ unique sheaf of rings $\mathcal{O}_X$ on $X = \mathrm{Spec}(A)$ such that $\mathcal{O}_X(D(f)) = A_f$ for all nonzero $f \in A$. Call $(X, \mathcal{O}_X)$ the <u>affine scheme</u> associated with A; $\mathcal{O}_X$ the <u>structure sheaf</u>.
- the <u>stalk</u> $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at x is

$$\coprod_{x \in U} \mathcal{O}_X(U) \Big/ \begin{array}{l} s \sim t, \quad s \in \mathcal{O}_X(U),\ t \in \mathcal{O}_X(V) \\ \text{if } \exists\, \underset{x}{W} \subset U \cap V \text{ with } s|_W = t|_W . \end{array}$$

- ringed spaces satisfying stalks are local rings is called a <u>locally ringed space</u>.
- morphism of ringed spaces: a pair $(\phi, \phi^{\#})$, $\phi : X \to Y$ continuous, $\phi^{\#} : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ (a pushforward for each open set); in the case of affine, $\varphi : A \to B$, A, B rings, induce $(\mathrm{Spec}(\varphi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A),\ \varphi)$.
- a <u>scheme</u> is a locally ringed space $(X, \mathcal{O}_X)$ having an open covering $\{U_i : i \in I\}$ such that $\forall i$ the locally ringed spaces $(U_i, \mathcal{O}_X|_{U_i})$ are isomorphic to affine schemes.
- examples:
  - k field, $\mathrm{Spec}(k) = \bullet$, stalk of structure sheaf at this point is k.
  - $\mathrm{Spec}\,\mathbb{Z}$ has one closed point corresponding to each prime (p) and a non-closed point corresponding to 0. Ring of sections of the structure sheaf over an open subset U is ring of rational numbers with denominator divisible only by primes outside U.

II. Many many conditions

- A scheme X is $\overset{\text{reduced}}{\text{integral}}$ if for all open subsets $U \subset X$ the ring $\mathcal{O}_X(U)$ has no $\overset{\text{nilpotents}}{\text{zero-divisors}}$.
- Recall: a topological space is irreducible if it cannot be expressed as a union of two proper closed subsets.
- (lem) A scheme X is integral iff it's reduced & its underlying space is irreducible.
- A integral scheme always has a unique point whose closure is the whole underlying space of the scheme "generic point". In affine: (0).
- The <u>dimension</u> $\dim X$ of a scheme X is the sup of integers for which there exists a

chain $Z_0 \subsetneq Z_1 \cdots \subsetneq Z_n$ of irreducible closed subsets properly contained in X.

- The dimension $\dim A$ of a ring is the dimension of $\mathrm{Spec}(A)$.

- A scheme is connected / quasi-compact if its underlying topological space is

  ↑ compactness without Hausdorff condition

- X <u>locally Noetherian</u> if stalks $\mathcal{O}_{X,p}$ of its structure sheaf are Noetherian local rings;

  + quasicompact $\Rightarrow X$ <u>Noetherian</u>.

- X <u>normal</u> if stalks $\mathcal{O}_{X,p}$ are integrally closed domains

  <u>regular</u> if $\mathcal{O}_{X,p}$ are regular local rings.

  A Noetherian local ring A with maximal ideal m and residue field $k = A/m$ is regular if $\dim_k m/m^2 = \dim A$.

- Ringed space given by U and $\mathcal{O}_U := (\mathcal{O}_X)|_U$ is also a scheme, the <u>open subscheme</u> associated with U. Morphism of schemes defined by topological inclusion $j: U \to X$ and the morphism of sheaves $\mathcal{O}_X \to j_* \mathcal{O}_U$ <u>open immersion</u>.

- A morphism $Z \to X$ of affine schemes is a closed immersion if it corresponds to $A \to A/I$ for some $I \subset A$. Generally, <u>closed immersion</u> if injective with closed image & restrictions to affine opens yields above.

II. Fibre products

- Categorical construction: $Y \times_X Z$, given $Y \xrightarrow{p} X$ and $Z \xrightarrow{q} X$, is the object with morphisms π_1, π_2 making

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow{p} & X \end{array}$$

commute & is universal with respect to this property.

- In Top, this is the ordinary Cartesian product with product topology.

- A <u>scheme over X</u> is a morphism $Y \longrightarrow X$ of schemes; a morphism of schemes over X is a morphism $Y \to Z$ compatible with projection onto X. Forms a category.

(Thm)
- Fibre products exist in the category of schemes over X! Call π_1 the base change of q via p, π_2 the base change of p via q.

- inclusion morphism $i_P : \mathrm{Spec}(\kappa(P)) \longrightarrow X$
- the fibre of $\phi : Y \to X$ at point P is the scheme $Y_P := Y \times_X \mathrm{Spec}(\kappa(P))$, the fibre product being taken with respect to maps ϕ, i_P.
- CAUTION: in general, the underlying topological space of a fibre product of schemes is NOT the topological fibre product of underlying topological space!
  - k field, k_s separable closure, $L|k$ separable extension, then $\mathrm{Spec}(L) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k_s) = \mathrm{Spec}(L \otimes_k k_s)$ is a finite disjoint union of copies of $\mathrm{Spec}(k_s)$, whereas topological fibre is just a point.
- diagonal map $\Delta : Y \to Y \times_X Y$ coming from a morphism of schemes $Y \to X$; induced by the identity map of Y in both coordinates.
- a morphism $Y \to X$ of schemes is <u>separated</u> if the diagonal map is a closed immersion.

☆ analogous to Hausdorff condition in topology.
- $\phi : Y \to X$ is <u>locally of finite type</u> if X has an afine open covering by subsets $U_i = \mathrm{Spec}(A_i)$ so that $\phi^{-1}(U_i)$ has an open covering $V_{ij} = \mathrm{Spec}(B_{ij})$ with finitely generated A_i- algebras B_{ij}. It is of <u>finite type</u> if there is such an open covering with finitely many V_{ij} for each i.
- A separated morphism $Y \to X$ of schemes is <u>proper</u> if it is of finite type and for every morphism $Z \to X$ the base change map $Y \times_X Z \to Z$ is a closed map (maps closed subsets to closed subsets.

IV. $\mathcal{O}_X$ - modules and quasi - coherence

- X scheme. A <u>sheaf of $\mathcal{O}_X$ - modules</u> is a sheaf of abelian groups $\mathcal{F}$ on X such that for each open $U \subset X$ the group $\mathcal{F}(U)$ is equipped with an $\mathcal{O}_X(U)$ - module structure $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ making the diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

commute for each inclusion of open sets $V \subset U$. If $\mathcal{F}(U)$ ideal in $\mathcal{O}_X(U)$, <u>sheaf of ideals</u>.
- $\phi : X \to Y$ morphism of schemes. On the level of structure sheaves $\phi^\# : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$, $\mathcal{O}_Y$ - module structure on $\phi_* \mathcal{O}_X$.
- kernel $\mathcal{I}$ of morphism $\phi^\# : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ is a sheaf of ideals on Y.

- lem. $X = \mathrm{Spec}(A)$ affine-scheme, M A-module. There is a unique $\mathcal{O}_X$-module $\tilde{M}$ satisfying $\tilde{M}(D(f)) = M \otimes_A A_f$ over each basic open set $D(f) \subset X$.
- X scheme. A <u>quasi-coherent sheaf</u> on X is an $\mathcal{O}_X$-module $\mathcal{F}$ for which there is open affine cover $\{U_i\}$ such that the restriction of $\mathcal{F}$ to each $U_i = \mathrm{Spec}(A_i)$ is isomorphic to an $\mathcal{O}_{U_i}$ module of the form $\tilde{M}_i$ with some A_i-module M_i. If each M_i is finitely generated over A_i, $\mathcal{F}$ is a <u>coherent sheaf</u>. $\mathcal{F}$ is <u>locally free</u> if M_i are free A-modules. Locally free sheaves of rank 1 are <u>invertible sheaves</u>.
- the functor $M \to \tilde{M}$ establishes an equivalence of categories between the category of A-modules and the category of quasi-coherent sheaves on X.
- a morphism of schemes $\phi: X \to Y$ is <u>affine</u> if Y has a covering by affine open subsets $U_i = \mathrm{Spec}\, A_i$ such that $\phi^{-1}(U_i)$ are all affine,
- lem. If $\phi: X \to Y$ is affine and $\mathcal{F}$ is a quasicoherent sheaf on X, then $\phi_* \mathcal{F}$ is a quasicoherent sheaf on Y. If $\mathcal{F}$ is coherent and ϕ is finite, then $\phi_* \mathcal{F}$ is coherent.