Galois 2023 Day 4

I. Galois covers

Recall: A cover of X is a space Y over X with $p: Y \to X$ such that $\forall x \in X$ $\exists$ open $V \ni x$ s.t. $p^{-1}(V)$ is a disjoint union of $\mathcal{U}_i$, each homeomorphic with V.

Def. Let G be a group acting continuous from the left on a topological space Y. The action of G is <u>even</u> if each point $y \in Y$ has some open neighborhood $\mathcal{U}$ such that the open sets $g\mathcal{U}$ are pairwise disjoint for all $g \in G$.

Lem. If G is a group acting evenly on a connected space Y, the projection $p_G: Y \to Y/G$ into a cover of Y/G.

Eg. The action of $\mathbb{Z}$ on $\mathbb{R}$ by translations ($x \mapsto x+n$ for $n \in \mathbb{Z}$) gives cover $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$, homeomorphic to a circle.

From here, fix space X <u>locally connected</u> (each point has basis of nbhd consisting of connected open subsets)

Cover $p: Y \to X$, $\mathrm{Aut}(Y|X)$ = automorphisms of Y as a space compatible with p:
$$\begin{tikzcd} Y \arrow[rr, "\sim"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}.$$

group w.r.t. composition

also acts on $p^{-1}(x)$

Prop.* Z connected top. space, $Z \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{p} X$ continuous maps s.t. $p \circ f = p \circ g$.

$\exists z \in Z$ with $f(z) = g(z) =: y \Rightarrow f = g$. Exercise!

Cor: An automorphism of a connected cover $p: Y \to X$ having a fixed point must be trivial.

Prop. If G group acting evenly on connected space Y, the automorphism group of the cover $p_G: Y \to G\backslash Y$ is precisely G.

Proof: $A := \mathrm{Aut}(Y \mid (G\backslash Y))$.

- G is a natural subgroup of A
- Take $\phi \in A$, look at its action on $y \in Y$. Fibres of p_G are orbits of G so $\exists g$ s.t. $\phi(y) = gy$.

  Apply cor. to $\phi \circ g^{-1}$ gives $\phi = g$.

Prop. If $p: Y \to X$ is a connected cover, the action of $\mathrm{Aut}(Y|X)$ is even. Exercise!

Notice p factors as $Y \to \mathrm{Aut}(Y|X)\backslash Y \xrightarrow{\bar{p}} X$, continuous.

Def. A cover $p: Y \to X$ <u>Galois</u> if Y is connected and the induced map $\bar{p}$ is a homeomorphism.

Fact: A connected cover $p: Y \to X$ is Galois iff $\mathrm{Aut}(Y|X)$ acts transitive on $\overset{\text{(one)}}{\text{each}}$ fibre of p.

Example: $X = \mathbb{R}^n/\mathbb{Z}^n$, fix $m > 1$ integer. Consider $X \xrightarrow{\cdot m} X$.

Galois cover with group $(\mathbb{Z}/m\mathbb{Z})^n$.

Thm. Let $p: Y \to X$ be a Galois cover. (1) For each subgroup H of $G = \mathrm{Aut}(Y|X)$ the projection p induces a natural map $\bar{p}_H : H\backslash Y \to X$ which turns $H\backslash Y$ into a cover of X.

(2) Conversely, if $Z \to X$ is a connected cover fitting into a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \arrow[d, "q"] \\ & X \end{tikzcd}$$

Then $f: Y \to Z$ is a Galois cover and actually $Z \cong H\backslash Y$ for the subgroup $H = \mathrm{Aut}(Y|Z)$ of G. The maps $H \mapsto H\backslash Y$, $Z \to \mathrm{Aut}(Y|Z)$, $Z \mapsto \mathrm{Aut}(Y|Z)$ induce a bijection between subgroups of G and intermediate covers Z. (3) The cover $q: Z \to X$ is Galois iff H is a normal subgroup of G, in which case $\mathrm{Aut}(Z|X) \cong G/H$.

Lem. Assume given a connected cover $q: Z \to X$ and a continuous map $f: Y \to Z$. If the composite $q \circ f : Y \to X$ is a cover, then so is $f: Y \to Z$. (Proof see book)

Proof of Thm:

(1) Since $H \subset \mathrm{Aut}(Y|X)$, p factors as $Y \xrightarrow{p_H} H\backslash Y \xrightarrow{\bar{p}_H} X$.

- $\bar{p}_H$ continuous since p, p_H are covers
- for sufficiently small V, $p^{-1}(V) \cong V \times F$: F discrete equipped with H-action
- $\bar{p}_H^{-1}(V) \subset H\backslash Y$ isomorphic to product of V and the discrete set of H-orbits of F. $\Rightarrow \bar{p}_H : H\backslash Y \to X$ cover.

(2)
- $f: Y \to Z$ is a cover, $H = \mathrm{Aut}(Y|Z)$ is a subgroup of G
- suffices to check H acts transitively on each fibre of f.
  - $z \in Z$, $y_1, y_2 \in f^{-1}(z)$. Both contained in fibre $p^{-1}(q(z))$, so since p Galois, $y_1 = \phi(y_2)$ with some $\phi \in G$.
  - apply prop ★, $g = f \circ \phi$, f, says $S = \{y \in Y : f(y) = f(\phi(y))\}$ is the whole Y.

    $\Rightarrow \phi \in H$.

(3)
- If H normal in G, G/H acts naturally on $Z = H\backslash Y$, preserving projection q. So we have injective group homom. $G/H \to \mathrm{Aut}(Z|X)$. But also $(G/H)\backslash Z \cong G\backslash Y \cong X$, so $G/H \cong \mathrm{Aut}(Z|X)$ and $q: Z \to X$ is Galois.

- Assume $Z \to X$ is Galois cover.

First show each element ϕ of $G = \mathrm{Aut}(Y|X)$ induces automorphism of Z over X: find $\psi : Z \to Z$ to make

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Y \\ f\downarrow & & \downarrow f \\ Z & \xrightarrow{\psi} & Z \\ q\downarrow & & \downarrow q \\ X & \xrightarrow{id} & X \end{array}$$

Commute.

Take $y \in Y$ with image $x = (q \circ f)(y)$ in X.

- $f(y)$ and $f(\phi(y))$ are in same fibre $q^{-1}(x)$ of q.
- Since $\mathrm{Aut}(Z|X)$ acts transitively on fibres of q, $\exists! \, \psi \in \mathrm{Aut}(Z|X)$ that $\psi(f(y)) = f(\phi(y))$
- Hence $\psi \circ f = f \circ \phi$ by ★.

- $\phi \mapsto \psi$ homom. $G \to \mathrm{Aut}(Z|X)$ whose kernel is $H = \mathrm{Aut}(Y|Z)$, hence normal! □

II. The monodromy action

Lem. Let $p: Y \to X$ be a cover, y a point of Y and $x = p(y)$.

1. Given a path $f: [0,1] \to X$ with $f(0) = x$, there is a unique path $\tilde{f}: [0,1] \to Y$ with $\tilde{f}(0) = y$ and $p \circ \tilde{f} = f$.
2. Second path $g: [0,1] \to X$ homotopic to f. Then $\tilde{g}: [0,1] \to Y$ with $\tilde{g}(0) = y$, $p \circ \tilde{g} = g$, we have $\tilde{f}(1) = \tilde{g}(1)$. (actually $\tilde{f}, \tilde{g}$ homotopic!)

(Proof involves chasing things down using compactness, see book.)

Monodromy action

Given $y \in p^{-1}(x)$ and $\alpha \in \pi_1(X, x)$ represented by a loop $f: [0,1] \to X$, define $\alpha y := \tilde{f}(1)$. Well-defined by lem; left action of $\pi_1(X, x)$ on $p^{-1}(x)$.

- Functor Fib_x from category of covers of X to category of sets equipped with a left $\pi_1(X,x)$-action: send cover $p: Y \to X$ to the fibre $p^{-1}(x)$.

Thm. X connected & locally simply connected with basepoint x.

Fib_x induces an equiv. of category of covers of X with the category of left $\pi_1(X, x)$ - sets.

Connected covers $\rightsquigarrow$ $\pi_1(X, x)$ - sets with transitive action

Galois covers $\rightsquigarrow$ coset spaces of normal subgroups.