Galois 2023 Day 2

I. Some Category Theory

- A <u>category</u> $\mathcal{C}$ consists of <u>objects</u> & <u>morphisms</u> between pairs of objects.

  Given objects A, B of $\mathcal{C}$, denote the set of morphisms from A to B $\mathrm{Hom}(A, B)$, which are subject to:

  - For all objects A, $id_A \in \mathrm{Hom}(A, A)$
  - Given $\psi \in \mathrm{Hom}(A, B)$, $\phi \in \mathrm{Hom}(B, C)$, there exists composition $\phi \circ \psi \in \mathrm{Hom}(A, C)$
  - Given $\phi \in \mathrm{Hom}(A, B)$, $\phi \circ id_A = id_B \circ \phi = \phi$
  - Given $\lambda \in \mathrm{Hom}(A, B)$, $\psi \in \mathrm{Hom}(B, C)$, $\phi \in \mathrm{Hom}(C, D)$, $(\phi \circ \psi) \circ \lambda = \phi \circ (\psi \circ \lambda)$.
- $\phi \in \mathrm{Hom}(A, B)$ is an <u>isomorphism</u> if there exists $\psi \in \mathrm{Hom}(B, A)$ with $\psi \circ \phi = id_A$, $\phi \circ \psi = id_B$; $\mathrm{Isom}(A, B)$
- <u>opposite category</u> $\mathcal{C}^{op}$: "category with the same objects and arrows reversed" ("preserves id & composition")
- <u>product</u> of categories $\mathcal{C}_1$, $\mathcal{C}_2$: $\mathcal{C}_1 \times \mathcal{C}_2$ where objects are pairs $(\underset{\in \mathcal{C}_1}{C_1}, \underset{\in \mathcal{C}_2}{C_2})$ and morphisms are pairs $(\underset{\in \mathcal{C}_1}{\phi_1}, \underset{\in \mathcal{C}_2}{\phi_2})$.
- $\mathcal{D}$ is a <u>subcategory</u> of $\mathcal{C}$ if it contain some objects and some morphisms of $\mathcal{C}$
  - <u>full</u> if for objects $A, B \in \mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Some useful examples:

- Sets : Objects are sets, morphisms are set-theoretic maps.
- Ab : Objects are abelian groups, morphisms are group homomorphisms.
- Top : Objects are topological groups, morphisms are continuous maps.

subcategories but not full

- <u>(covariant) functor</u> F between $\mathcal{C}_1$, $\mathcal{C}_2$ consists of
  - $A \mapsto F(A)$ on objects
  - $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ on sets of morphism, preserve id and composition.
- <u>contravariant functor</u> from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a funcor from $\mathcal{C}_1$ to $\mathcal{C}_2^{op}$.

Examples of functors:

- id$_{\mathcal{C}}$ - leaves all objects & morphisms fixed.
- Fix object A of $\mathcal{C}$; covariant $\mathrm{Hom}(A, \ )$ from $\mathcal{C}$ to Sets sending $B \mapsto \mathrm{Hom}(A, B)$,

  contravariant $\mathrm{Hom}(\ , A)$ $B \mapsto \mathrm{Hom}(B, A)$

and $\phi: B \to C$ to $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ as sets induced by composition with ϕ.

$\mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A)$ (Questions ?)

- $\mathcal{C}_1 \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{C}_2$ two functors, a <u>morphism of functors</u> Φ is a collection of morphisms in $\mathcal{C}_2$ $\Phi_A : F(A) \to G(A)$ for each object A such that

$$\begin{tikzcd} F(A) \arrow[r, "\Phi_A"] \arrow[d, "F(\phi)"'] \arrow[dr, phantom, "\circlearrowright"] & G(A) \arrow[d, "G(\phi)"] \\ F(B) \arrow[r, "\Phi_B"'] & G(B) \end{tikzcd}.$$

Makes functors a category. The morphism Φ is an isomorphism if each Φ_A is an isomorphism, write $F \cong G$.

"natural transformation"

- $\mathcal{C}_1, \mathcal{C}_2$ <u>equivalent</u> if $\exists$ $F: \mathcal{C}_1 \to \mathcal{C}_2$, $G: \mathcal{C}_2 \to \mathcal{C}_1$ and isomorphisms $\Phi: F \circ G \xrightarrow{\sim} id_{\mathcal{C}_2}$, $\Psi: G \circ F \xrightarrow{\sim} id_{\mathcal{C}_1}$. <u>Quasi - inverse</u>; if $F \circ G = id_{\mathcal{C}_2}$, $G \circ F = id_{\mathcal{C}_1}$, <u>isomorphic</u>.

(anti-equivalent, anti-isomorphic if $\mathcal{C}_1$ is equivalent, isomorphic to $\mathcal{C}_2^{op}$.)

- Note: equivalence relation : check reflexive, symmetric, transitive.
- $F: \mathcal{C}_1 \to \mathcal{C}_2$ <u>faithful</u> if for any objects A, B of $\mathcal{C}_1$ the map of sets $F_{AB} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ induced by F is surjective; <u>fully faithful</u> if all maps F_{AB} are bijective.
- $F: \mathcal{C}_1 \to \mathcal{C}_2$ <u>essentially surjective</u> if every object of $\mathcal{C}_2$ is isomorphic to some object of the form $F(A)$.

<u>Lem</u> Two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent iff $\exists$ $F: \mathcal{C}_1 \to \mathcal{C}_2$ which is fully faithful and essentially surjective.

Proof. ($\Leftarrow$) For each object V of $\mathcal{C}_2$, there is isomorphism $i_V : F(A) \xrightarrow{\sim} V$.

Define $G: \mathcal{C}_2 \to \mathcal{C}_1$ by sending V to A and morphism $\phi: V \to W$ to $G(\phi) = F_{AB}^{-1}(i_W^{-1} \circ \phi \circ i_V)$ where $B = G(W)$. By construction $\Phi : F \circ G \xrightarrow{\sim} id_{\mathcal{C}_2}$. (Questions ?)

Now construct isomorphism $\Psi: G \circ F \xrightarrow{\sim} id_{\mathcal{C}_1}$: need functorial maps $\Psi_A : G(F(A)) \xrightarrow{\sim} A$. By fully faithfulness of F, suffices to construct $F(\Psi_A) : F(G(F(A))) \to F(A)$; take $F(\Psi_A)$ to be the unique preimage of $id_{F(A)}$ by $\Phi_{F(A)}$. Similar construction yields inverse $A \to G(F(A))$. Hence Ψ isom. of functors.

($\Rightarrow$) Assume we have $G: \mathcal{C}_2 \to \mathcal{C}_1$ and isomorphism of functors $\Phi : F \circ G \xrightarrow{\sim} id_{\mathcal{C}_2}$ and $\Psi: G \circ F \xrightarrow{\sim} id_{\mathcal{C}_1}$.

Essential surj. : given C of $\mathcal{C}_2$, it is isomorphic to $F(G(C))$ via Φ.

Fully faithful:

$$\mathrm{Hom}(A, B) \xrightarrow{F_{AB}} \mathrm{Hom}(F(A), F(B)) \xrightarrow{G_{F(A),F(B)}} \mathrm{Hom}(G(F(A)), G(F(B))) \xrightarrow{\Psi} \mathrm{Hom}(A, B).$$

- $F : \mathcal{C} \to$ Sets is <u>representable</u> if $\exists$ object $C \in \mathcal{C}$ and an isomorphism of functors $F \cong \mathrm{Hom}(C, \ )$.

"representing object"

<u>Yoneda Lemma</u> If F, G are functors $\mathcal{C} \to$ Sets represented by objects C and D resp., every morphism $\Phi : F \to G$ is induced by a unique morphism $D \to C$.

Proof. Rewrite $\Phi : F(C) \to G(C)$ as $\mathrm{Hom}(C, C) \to \mathrm{Hom}(D, C)$.

Image of $id_C \in \mathrm{Hom}(C, C)$ identifies $p : D \to C$; this induces Φ :

for object A each element of $F(A) \cong \mathrm{Hom}(C, A)$ identifies with morphism $\phi : C \to A$ which is the image of $id_C \in \mathrm{Hom}(C, C) \cong F(C)$ via $F(\phi)$. As Φ is a morphism of functors, $\Phi_A(\phi) = G(\phi)(p)$ which under the isomorphism corresponds exactly to $\phi \circ p$.

Cor. The representing object of a representable functor F is unique up to unique isomorphism.

II. Intro to Sheaves

- X top. space. A <u>presheaf</u> (of sets) $\mathcal{F}$ on X is the data :
  - For each open $U \subset X$, a set $\mathcal{F}(U)$
  - $V \subset U$ a map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$;

    ρ_{UU} identity, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for $W \subset V \subset U$.
- Elements of $\mathcal{F}(U)$ called <u>sections</u> of $\mathcal{F}$ over U.
- Can have presheaf of groups, ab. groups, rings etc. with $\mathcal{F}(U)$ being those things and maps homomorphisms.
- presheaves on fixed top. space form a category
- example: cont. real-valued functions defined locally on open subsets of X. ρ_{UV} given by restriction of functions.
- A presheaf $\mathcal{F}$ is a <u>sheaf</u> if it satisfies
  - Given nonempty U & covering $\{U_i\}$, if $s, t \in \mathcal{F}(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.
  - Given $\{U_i\}$, system of sections $\{s_i \in \mathcal{F}(U_i)\}$, and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ whenever $U_i \cap U_j \neq \phi$, $\exists\ s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i ; by above, s unique.