The Proof of the Nilpotence Theorem

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August 11, 2022

Outline

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Statements

Theorem (Nilpotence – Strong Form)

Let R be a ring spectrum, and $h\colon \pi_* R \to MU_* R$ be the Hurewicz map. Then $\ker h$ consists of nilpotent elements.

Theorem (Nilpotence – Smash Product Form)

Let $f : F \to X$ be a map from a finite spectrum to an arbitrary one. If $1_{MU} \wedge f$ is null, then f is smash nilpotent; i.e. $f^{\wedge k} = 0$ for large enough k.

Reducing to Weak Nilpotence

Theorem (Nilpotence – Weak Form)

Let R be a connective associative ring spectrum of finite type. Then $\ker h$ consists of nilpotent elements.

This implies the smash product form!

Proof.

Reduce to the case $F = S^0$ by Spanier-Whitehead duality. Since MU is a ring spectrum, $1_{MU} \wedge f$ is null iff $S^0 \xrightarrow{f} X \to MU \wedge X$ is. As X is a colimit of finite spectra, we can reduce to a finite spectrum. Replacing the target by its free monoid lets us apply weak nilpotence; knowing that $1_{MU} \wedge f$ is null, we conclude f is smash nilpotent! □

The Key Characters

Define ring spectra $X(n)$ by taking the Thom spectra of $\Omega SU(n) \to \Omega SU \to BU$. Then $X(1) = S^0$ and $MU = \varinjlim X(n)$.

Theorem

For R satisfying weak nilpotence conditions, and $\alpha \in \pi_* R$, if $X(n+1)_* \alpha$ is zero, then $X(n)_* \alpha$ is nilpotent.

Proof.

If $h(\alpha) = 0$, then because $MU = \varinjlim X(n)$, $X(n+1)_* \alpha = 0$ for sufficiently large n. By the above, we conclude that $X(1)_* \alpha = \alpha = 0$. □

Plan of Attack

As it turns out, $\langle X(n) \rangle > \langle X(n+1) \rangle$, so we'll need to interpolate further via spectra G_k.

- Step I: Show that if $h(n+1)_*(\alpha) = 0$, then $G_k \wedge \alpha^{-1}R = 0$ for large enough k.
- Steps II: Show that $\langle G_k \rangle = \langle X(n) \rangle$ for all k.
- Observe that for E a ring, $E \wedge \alpha^{-1}R = 0$ implies E-Hurewicz image of α is nilpotent. (This is easy.)

Steps I and II are *very* technical!

Outline

Properties of $X(n)$

Definition

The *James construction* JX on a based complex X is the free unital monoid on X. It admits a filtration $J_k X = \mathrm{im}(X^{\times k} \to JX)$, and $JX = \varinjlim J_k X$. In fact, $JX \simeq \Omega\Sigma X$.

In particular, we can interpolate $X(n+1)$ to $X(n)$ via the James filtration associated to the fiber sequence

$$\Omega SU(n) \to \Omega SU(n+1) \to \Omega S^{2n+1}.$$

Properties of $X(n)$

Define B_k via the pullback

$$\begin{array}{ccc} B_k & \longrightarrow & J_k S^{2n} \\ \downarrow & & \downarrow \\ \Omega SU(n+1) & \longrightarrow & \Omega S^{2n+1} \end{array}$$

Let F_k be Thom spectra of the lower composite, and set $G_k := F_{p^k - 1}$.

AHSS computations will show that $X(n+1)_* X(n+1)$ and $X(n+1)_* G_k$ are flat over $X(n+1)_*$.

Vanishing Lines in ANSS

Want $G_k \wedge \alpha^{-1}R = 0$. Let's force things in $X(n+1)$-ANSS to vanish!

Theorem

Let M be a connective $X(n+1)_*X(n+1)$-comodule of finite type. Then

$$E^{s,t} = \mathrm{Ext}^{s,t}_{X(n+1)_*X(n+1)}\left(X(n+1)_*, X(n+1)_* G_k \otimes_{X(n+1)_*} M\right),$$

has a vanishing line of slope going to zero as k increases. Specifically, for any small slope $1/m$, there exists k and c such that $E^{s,t} = 0$ whenever $t - s < ms - c$.

Proof of Step I

Finally: $\pi_* R$ acts on $\pi_* G_k \wedge R$. We want to show that for every $\beta \in \pi_* G_k \wedge R$, there exists an m such that $\beta\alpha^m = 0$.

Consider the two $X(n+1)$-ANSS E_2's

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{X(n+1)_* X(n+1)}(X(n+1)_*, X(n+1)_* R) \implies \pi_* R,$$

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{X(n+1)_* X(n+1)}(X(n+1)_*, X(n+1)_* G_k \wedge R) \implies \pi_* G_k \wedge R.$$

Because $X(n+1)_*\alpha = 0$, it is detected by some $a \in \mathrm{Ext}^{s,t}(R)$, for $s > 0$. Choose k so that the Ext for $M = X(n+1)_* R$ has a vanishing line of slope less than $s/(t-s)$.

Proof of Step I

By flatness,

$$X(n+1)_*G_k \wedge R = X(n+1)_*G_k \otimes_{X(n+1)_*} X(n+1)_*R,$$

so the E_2 of the ANSS for $\pi_*G_k \wedge R$ has such a vanishing line!

Let $\beta \in \pi_*G_k \wedge R$ be detected by some $b \in E_2^{u,v}$. Then, if $\beta\alpha^m = 0$, it must be detected by an element in $E_2^{u+ms+j,v+mt+j}$ for $j \geq 0$. By our vanishing line, this will be zero for m large. Thus, $\beta\alpha^m = 0$, and we are done!

Outline

The Strategy

We'll start by constructing a self-map b of G_k such that its cofiber is G_{k+1}.

Theorem

Given X and self-map $g\colon \Sigma^d X \to X$, then $\langle X\rangle = \left\langle \operatorname{colim} \Sigma^{-id} X \right\rangle \vee \langle C(g)\rangle$.

Thus, we'll end up with $\langle G_k\rangle = \langle G_{k+1}\rangle \vee \left\langle b^{-1} G_k \right\rangle$. Then, we show $b^{-1} G_k = 0$ and we're done!

Constructing the Self-map

We are given $\xi \colon E_{p=1} \to BU$, and suppose we have a fibration $E_{p-1} \to J_{p-1}S^{2m}$. For $0 \leq q \leq p-1$, let E_q be the pullback

$$\begin{array}{ccc} E_q & \longrightarrow & E_{p-1} \\ \downarrow & & \downarrow \\ J_q S^{2m} & \longrightarrow & J_{p-1}S^{2m} \end{array}$$

Iterating, we get a filtration $E_0 \subset E_1 \subset \ldots \subset E_{p-1}$. This gives a filtration of E^{ξ}_{p-1} by the E^{ξ}_q.

Constructing the Self-map

From here, one can define equivalences $\theta_{p-1}\colon E^\xi_{p-1}/E^\xi_{p-2} \to \Sigma^{2m(p-1)} E^\xi_0$ and $\theta_1\colon E^\xi_{p-1}/E^\xi_0 \to \Sigma^{2m} E^\xi_{p-2}$.

Define b as

$$\Sigma^{2mp-2} E^\xi_0 \xrightarrow{\theta_{p-1}^{-1}} \Sigma^{2m-2} E^\xi_{p-1}/E^\xi_{p-2} \longrightarrow \Sigma^{2m-1} E^\xi_{p-2}$$

$$\xrightarrow{\theta_1^{-1}} \Sigma^{-1} E^\xi_{p-1}/E^\xi_0 \longrightarrow E^\xi_0$$

Now, we get $\langle E^\xi_0 \rangle = \langle E^\xi_{p-1} \rangle \vee \langle b^{-1} E^\xi_0 \rangle$.

Constructing the Self-map

Lemma

There are (p-local) fiber sequences

$$J_{p^k-1}S^{2n} \to JS^{2n} \to JS^{2np^k},$$

and for $m > 1$,

$$J_{p^k-1}S^{2n} \to J_{mp^k-1}S^{2n} \to J_{m-1}S^{2np^k}.$$

Specialize to B_j and we get a fiber sequence

$$B_{p^k-1} \to B_{p^{k+1}-1} \xrightarrow{q} J_{p-1}S^{2p^k n}$$

This gives $E_0^\xi = G_k$ and $E_{p-1}^\xi = G_{k+1}$, as well as the self-map $b\colon \Sigma^{2np^{k+1}-2}G_k \to G_k$.

Finishing Step II

Consider the following diagram of fiber sequences:

$$\begin{array}{ccccc}
\Omega^2 S^{2np^k+1} & \longrightarrow & B_{p^k-1} & \longrightarrow & \Omega SU(n+1) \\
\| & & \downarrow & & \downarrow \\
\Omega^2 S^{2np^k+1} & \longrightarrow & J_{p^k-1} S^{2n} & \longrightarrow & \Omega S^{2n+1}
\end{array}$$

Thomifying, get an action of $\Sigma^\infty_+ \Omega^2 S^{2np^k+1}$ on G_k. Recall Snaith's splitting

$$\Sigma^\infty_+ \Omega^2 S^{2np^k+1} \simeq \bigvee_{i \geq 0} D_i,$$

$$D_i = \left(\Sigma^{(2np^k-1)i} \Sigma^\infty_+ C^{(2)}(i) \right)_{h\Sigma_i}.$$

Finishing Step II

Tracing through this, we get a map

$$f\colon \Sigma^{2np^{k+1}-2} G_k \to D_p \wedge G_k \to \Sigma^\infty_+ \Omega^2 S^{2np^k+1} \wedge G_k \to G_k$$

which is multiplication by some element β. In addition, $G_k \to f^{-1} G_k$ factors through $G_k \wedge H\mathbb{F}_p$.

This is the same as our map b. One computes that $B_{p^k-1} \to B_{p^{k+1}-1}$ is injective in $H\mathbb{F}_p$-homology, so $H\mathbb{F}_{p_*} b = 0$. Thus, the identity map $b^{-1} G_k \to b^{-1} G_k$ factors through $b^{-1} G_k \wedge H\mathbb{F}_p$, which is zero, so $b^{-1} G_k = 0$ and we're done.

Outline

What Just Happened?

That was complicated. Stepping down from $X(n+1)$ to $X(n)$ is hard, so something is special with $\langle G_k \rangle = \langle G_{k+1} \rangle$.

Need to use the map b to figure out this speciality. No alternative to step II of this proof exists.

This gives a recipe for how MU "sees" the stable homotopy category, and most of Ravenel conjectures hinge on the Nilpotence theorem

Axiomatic Nilpotence

In full generality, this proof tells us what spectra "detect nilpotence."

Theorem

Let $R \to T$ be a map of ring spectra with R detecting nilpotence. If T is a filtered colimit of spectra G_k such that

- *T-ANSS for $G_k \wedge R$ has vanishing lines of arbitrarily small slopes,*
- *$\langle G_k \rangle = \langle R \rangle$ for all k,*

then T detects nilpotence.

The sequel paper tries to characterize these spectra via Morava K-theory.

References

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