# The Proof of the Nilpotence Theorem

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Proof of Nilpotence

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## Outline

## Statement and Outline

2 Step I: Vanishing Lines

- 3 Step II: Thom Spectra
- 4 Vistas and Outlooks

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## Statements

## Theorem (Nilpotence – Strong Form)

Let R be a ring spectrum, and  $h: \pi_* R \to MU_* R$  be the Hurewicz map. Then ker h consists of nilpotent elements.

#### Theorem (Nilpotence – Smash Product Form)

Let  $f: F \to X$  be a map from a finite spectrum to an arbitrary one. If  $1_{MU} \wedge f$  is null, then f is smash nilpotent; i.e.  $f^{\wedge k} = 0$  for large enough k.

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# Reducing to Weak Nilpotence

## Theorem (Nilpotence – Weak Form)

Let R be a connective associative ring spectrum of finite type. Then ker h consists of nilpotent elements.

This implies the smash product form!

#### Proof.

Reduce to the case  $F = S^0$  by Spanier-Whitehead duality. Since MU is a ring spectrum,  $1_{MU} \wedge f$  is null iff  $S^0 \xrightarrow{f} X \to MU \wedge X$  is. As X is a colimit of finite spectra, we can reduce to a finite spectrum. Replacing the target by its free monoid lets us apply weak nilpotence; knowing that  $1_{MU} \wedge f$  is null, we conclude f is smash nilpotent!

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# The Key Characters

# Define ring spectra X(n) by taking the Thom spectra of $\Omega SU(n) \rightarrow \Omega SU \rightarrow BU$ . Then $X(1) = S^0$ and $MU = \lim_{n \to \infty} X(n)$ .

#### Theorem

For R satisfying weak nilpotence conditions, and  $\alpha \in \pi_*R$ , if  $X(n+1)_*\alpha$  is zero, then  $X(n)_*\alpha$  is nilpotent.

#### Proof.

If  $h(\alpha) = 0$ , then because  $MU = \lim_{n \to \infty} X(n)$ ,  $X(n+1)_*\alpha = 0$  for sufficiently large *n*. By the above, we conclude that  $X(1)_*\alpha = \alpha = 0$ .

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# Plan of Attack

As it turns out,  $\langle X(n) \rangle > \langle X(n+1) \rangle$ , so we'll need to interpolate further via spectra  $G_k$ .

- Step I: Show that if  $h(n+1)_*(\alpha) = 0$ , then  $G_k \wedge \alpha^{-1}R = 0$  for large enough k.
- Steps II: Show that  $\langle G_k \rangle = \langle X(n) \rangle$  for all k.
- Observe that for E a ring,  $E \wedge \alpha^{-1}R = 0$  implies E-Hurewicz image of  $\alpha$  is nilpotent. (This is easy.)

Steps I and II are very technical!

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# Outline



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# Properties of X(n)

#### Definition

The James construction JX on a based complex X is the free unital monoid on X. It admits a filtration  $J_k X = im(X^{\times k} \to JX)$ , and  $JX = \lim_{K \to K} J_k X$ . In fact,  $JX \simeq \Omega \Sigma X$ .

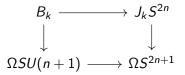
In particular, we can interpolate X(n + 1) to X(n) via the James filtration associated to the fiber sequence

$$\Omega SU(n) 
ightarrow \Omega SU(n+1) 
ightarrow \Omega S^{2n+1}$$

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Properties of X(n)

Define  $B_k$  via the pullback



Let  $F_k$  be Thom spectra of the lower composite, and set  $G_k := F_{p^k-1}$ .

AHSS computations will show that  $X(n+1)_*X(n+1)$  and  $X(n+1)_*G_k$  are flat over  $X(n+1)_*$ .

# Vanishing Lines in ANSS

Want  $G_k \wedge \alpha^{-1}R = 0$ . Let's force things in X(n+1)-ANSS to vanish!

#### Theorem

Let M be a connective  $X(n+1)_*X(n+1)$ -comodule of finite type. Then

$$E^{s,t} = \operatorname{Ext}_{X(n+1)_*X(n+1)}^{s,t} \left( X(n+1)_*, X(n+1)_* G_k \otimes_{X(n+1)_*} M \right),$$

has a vanishing line of slope going to zero as k increases. Specifically, for any small slope 1/m, there exists k and c such that  $E^{s,t} = 0$  whenever t - s < ms - c.

## Proof of Step I

Finally:  $\pi_*R$  acts on  $\pi_*G_k \wedge R$ . We want to show that for every  $\beta \in \pi_*G_k \wedge R$ , there exists an *m* such that  $\beta \alpha^m = 0$ .

Consider the two X(n+1)-ANSS  $E_2$ 's

$$E_2^{s,t} = \mathsf{Ext}_{X(n+1)_*X(n+1)}^{s,t} (X(n+1)_*, X(n+1)_*R) \implies \pi_*R,$$

$$E_2^{s,t} = \operatorname{Ext}_{X(n+1)_*X(n+1)}^{s,t} (X(n+1)_*, X(n+1)_*G_k \wedge R) \implies \pi_*G_k \wedge R.$$

Because  $X(n+1)_*\alpha = 0$ , it is detected by some  $a \in \operatorname{Ext}^{s,t}(R)$ , for s > 0. Choose k so that the Ext for  $M = X(n+1)_*R$  has a vanishing line of slope less than s/(t-s).

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# Proof of Step I

By flatness,

$$X(n+1)_*G_k \wedge R = X(n+1)_*G_k \otimes_{X(n+1)_*} X(n+1)_*R,$$

so the  $E_2$  of the ANSS for  $\pi_*G_k \wedge R$  has such a vanishing line!

Let  $\beta \in \pi_*G_k \wedge R$  be detected by some  $b \in E_2^{u,v}$ . Then, if  $\beta \alpha^m = 0$ , it must be detected by an element in  $E_2^{u+ms+j,v+mt+j}$  for  $j \ge 0$ . By our vanishing line, this will be zero for *m* large. Thus,  $\beta \alpha^m = 0$ , and we are done!

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# The Strategy

We'll start by constructing a self-map b of  $G_k$  such that its cofiber is  $G_{k+1}$ .

#### Theorem

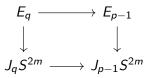
Given X and self-map  $g \colon \Sigma^d X \to X$ , then  $\langle X \rangle = \langle \operatorname{colim} \Sigma^{-id} X \rangle \lor \langle C(g) \rangle$ .

Thus, we'll end up with  $\langle G_k \rangle = \langle G_{k+1} \rangle \vee \langle b^{-1}G_k \rangle$ . Then, we show  $b^{-1}G_k = 0$  and we're done!

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# Constructing the Self-map

We are given  $\xi: E_{p=1} \to BU$ , and suppose we have a fibration  $E_{p-1} \rightarrow J_{p-1}S^{2m}$ . For  $0 \le q \le p-1$ , let  $E_q$  be the pullback



Iterating, we get a filtration  $E_0 \subset E_1 \subset \ldots \subset E_{p-1}$ . This gives a filtration of  $E_{p-1}^{\xi}$  by the  $E_q^{\xi}$ .

# Constructing the Self-map

From here, one can define equivalences  $\theta_{p-1} \colon E_{p-1}^{\xi}/E_{p-2}^{\xi} \to \Sigma^{2m(p-1)}E_0^{\xi}$ and  $\theta_1 \colon E_{p-1}^{\xi}/E_0^{\xi} \to \Sigma^{2m}E_{p-2}^{\xi}$ .

Define b as

$$\Sigma^{2mp-2} E_0^{\xi} \xrightarrow{\theta_{p-1}^{-1}} \Sigma^{2m-2} E_{p-1}^{\xi} / E_{p-2}^{\xi} \longrightarrow \Sigma^{2m-1} E_{p-2}^{\xi}$$
$$\xrightarrow{\theta_1^{-1}} \Sigma^{-1} E_{p-1}^{\xi} / E_0^{\xi} \longrightarrow E_0^{\xi}$$

Now, we get  $\langle E_0^{\xi} \rangle = \langle E_{p-1}^{\xi} \rangle \vee \langle b^{-1} E_0^{\xi} \rangle.$ 

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# Constructing the Self-map

#### Lemma

There are (p-local) fiber sequences

$$J_{p^k-1}S^{2n} \to JS^{2n} \to JS^{2np^k},$$

and for m > 1,

$$J_{p^{k}-1}S^{2n} \to J_{mp^{k}-1}S^{2n} \to J_{m-1}S^{2np^{k}}$$

Specialize to  $B_j$  and we get a fiber sequence

$$B_{p^k-1} o B_{p^{k+1}-1} \xrightarrow{q} J_{p-1}S^{2p^kn}$$

This gives  $E_0^{\xi} = G_k$  and  $E_{p-1}^{\xi} = G_{k+1}$ , as well as the self-map  $b \colon \Sigma^{2np^{k+1}-2}G_k \to G_k.$ 

# Finishing Step II

Consider the following diagram of fiber sequences:

Thomifying, get an action of  $\Sigma^{\infty}_{+}\Omega^2 S^{2np^k+1}$  on  $G_k$ . Recall Snaith's splitting

$$\Sigma^{\infty}_{+} \Omega^2 S^{2np^k+1} \simeq \bigvee_{i \ge 0} D_i,$$
 $D_i = \left( \Sigma^{(2np^k-1)i} \Sigma^{\infty}_{+} C^{(2)}(i) 
ight)_{h\Sigma}$ 

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# Finishing Step II

Tracing through this, we get a map

$$f: \Sigma^{2np^{k+1}-2}G_k \to D_p \wedge G_k \to \Sigma^{\infty}_+\Omega^2 S^{2np^k+1} \wedge G_k \to G_k$$

which is multiplication by some element  $\beta$ . In addition,  $G_k \to f^{-1}G_k$  factors through  $G_k \wedge HF_p$ .

This is the same as our map *b*. One computes that  $B_{p^{k}-1} \rightarrow B_{p^{k+1}-1}$  is injective in  $HF_{p}$ -homology, so  $HF_{p_{*}}b = 0$ . Thus, the identity map  $b^{-1}G_{k} \rightarrow b^{-1}G_{k}$  factors through  $b^{-1}G_{k} \wedge HF_{p}$ , which is zero, so  $b^{-1}G_{k} = 0$  and we're done.

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# What Just Happened?

That was complicated. Stepping down from X(n+1) to X(n) is hard, so something is special with  $\langle G_k \rangle = \langle G_{k+1} \rangle$ .

Need to use the map b to figure out this speciality. No alternative to step II of this proof exists.

This gives a recipe for how MU "sees" the stable homotopy category, and most of Ravenel conjectures hinge on the Nilpotence theorem

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# Axiomatic Nilpotence

In full generality, this proof tells us what spectra "detect nilpotence."

#### Theorem

Let  $R \to T$  be a map of ring spectra with R detecting nilpotence. If T is a filtered colimit of spectra  $G_k$  such that

• *T*-ANSS for  $G_k \wedge R$  has vanishing lines of arbitrarily small slopes,

• 
$$\langle G_k \rangle = \langle R \rangle$$
 for all k,

then T detects nilpotence.

The sequel paper tries to characterize these spectra via Morava K-theory.

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