The Ravenel Conjectures and the Chromatic Decomposition

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Outline

Bousfield Classes

Recall Bousfield localization at a homology theory E; we have a functor $L_E \colon \mathbf{SHC} \to \mathbf{SHC}$ and a universal natural E-equivalence $\eta \colon \mathrm{id} \implies L_E$. How do we compare different localizations?

Definition

Two homology theories E, F are *Bousfield equivalent* if $E_*X = 0 \iff F_*X = 0$. Equivalence classes are denoted $\langle E \rangle$.

1. If $E_*X = 0 \implies F_*X = 0$, then we write $\langle E \rangle \geq \langle F \rangle$.
2. Write $\langle E \rangle \vee \langle F \rangle = \langle E \vee F \rangle$ and $\langle E \rangle \wedge \langle F \rangle = \langle E \wedge F \rangle$.
3. Observe that $\langle 0 \rangle \leq \langle E \rangle \leq \langle S^0 \rangle$ for all E.

Some Properties

Theorem

$\langle E\rangle = \langle F\rangle$ iff $L_E = L_F$.
If $\langle E\rangle \leq \langle F\rangle$, then $L_E L_F = L_E$ and there is a natural transformation $L_F \implies L_E$.

Theorem

For all $n \in \mathbb{Z}$, $\langle X\rangle = \langle \Sigma^n X\rangle$.
If $X \to Y \to Z \to \Sigma X$ is a cofiber sequence, then $\langle X\rangle, \langle Y\rangle, \langle Z\rangle$ is $\leq$ the wedge of the other two.

Some More Properties

Definition

Let R be a ring spectrum, and $\alpha \in \pi_* R$. This gives a self-map

$$\Sigma^{|\alpha|} R = S^{|\alpha|} \wedge R \xrightarrow{\alpha \wedge 1} R \wedge R \xrightarrow{\mu} \mathrm{R},$$

which we'll denote α. Write R/α for the cofiber, and $\alpha^{-1} R$ for the colimit

$$\text{hocolim}\left[R \xrightarrow{\alpha} \Sigma^{-|\alpha|} R \xrightarrow{\alpha} \Sigma^{-2|\alpha|} R \xrightarrow{\alpha} \cdots\right]$$

Theorem

Given X and self-map $g \colon \Sigma^d X \to X$, then $\langle X \rangle = \left\langle \operatorname{colim} \Sigma^{-id} X \right\rangle \vee \langle Cg \rangle$. In particular, $\langle R \rangle = \left\langle \alpha^{-1} R \right\rangle \vee \langle R/\alpha \rangle$.

Outline

A Little Motivation

Definition

Fix a prime p. The Brown-Peterson spectrum BP is a summand of MU localized at p with homotopy groups

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots] \text{ with } |v_n| = 2(p^n - 1).$$

Remark

In fact, BP_* has a similar relation to FGLs as MU_*; it supports a universal p-typical formal group law!

Theorem (ANSS)

There is a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*X) \implies \pi_* L_{BP} X = \pi_* X_{(p)}.$$

BP and Friends

Definition

For each $n \geq 0$, we have spectra given by

$$E(n) = v_n^{-1} BP/(v_{n+1}, v_{n+2}, \ldots),$$

$$K(n) = v_n^{-1} BP/(p, v_1, v_2, \ldots, v_{n-1}, v_{n+1}, v_{n+2}, \ldots).$$

In particular, $E(0) = K(0) = H\mathbb{Q}$, and $E(1)$ is the Adams summand of p-local complex K-theory, $L_{(p)}K$.

Theorem

Let $I_n = (p, v_1, v_2, \ldots, v_{n-1})$. *We have that*

1. $\langle v_n^{-1} BP/I_n \rangle = \langle K(n) \rangle$,
2. $\langle v_n^{-1} BP \rangle = \langle E(n) \rangle$,
3. $\langle E(n) \rangle = \bigvee_{i=0}^{n} \langle K(n) \rangle$.

Interpolating BP and S

BP sees everything about $\mathbf{SHC}_{(p)}$. Before we go on, what does it miss?

Theorem

There exist Thom spectra $X(n)$ for $n \geq 0$ satisfying

$$\left\langle S^0_{(p)} \right\rangle = \langle X(0) \rangle > \langle X(1) \rangle > \langle X(2) \rangle > \ldots > \langle BP \rangle .$$

Sketch.

We have $\Omega SU \simeq BU$ by Bott Periodicity, and there is a natural map $SU(p^n) \to SU$, so we can define $X(n)$ to be the (localized) Thom spectra of the bundle classified by $\Omega SU(p^n) \to BU$. Showing that $\langle X(n) \rangle > \langle X(n+1) \rangle$ is a computation with the Adams spectral sequence (globally, we interpolate using the James filtration). This map is a 2-fold loop map, so $X(n)$ is a ring spectrum. □

Outline

The Chromatic Tower

Because $\langle E(n)\rangle \geq \langle E(n-1)\rangle$, we have natural transformations $L_n \to L_{n-1}$, so for any spectrum X, we get a tower

$$\cdots \to L_nX \to L_{n-1}X \to \cdots \to L_2X \to L_1X \to L_0X = X_{\mathbb{Q}}.$$

Conjecture (Chromatic Convergence)

If X is a p-local finite spectrum, then $L_\infty X := \varprojlim L_nX$ is equivalent to X.

In fact, by the chromatic fracture square, the difference between L_n and L_{n-1} is measured by $L_{K(n)}$. So, we can reconstruct this tower from the "gluing data."

Connections with Algebra

Write $\text{Ext}(BP_*)$ for the E_2 page for $X = S^0$. Set $N^0 = BP_*$, $M^n = v_n^{-1}N^n$, and inductively define N^n by

$$0 \to N^n \to M^n \to N^{n+1} \to 0.$$

Splice these into the *chromatic resolution*:

$$0 \to BP_* \to M_0 = p^{-1}BP_* \to M_1 \to M_2 \to \dots.$$

The associated *chromatic spectral sequence* is

$$E_1^{s,t} = \text{Ext}^t_{BP_*BP}(BP_*, M^s) \implies \text{Ext}(BP_*).$$

Connections with Algebra

Why does this help? We can understand $\mathrm{Ext}\, v_n^{-1}BP_*/I_n$ via the *Morava stabilizer group*. It's v_n-periodic, i.e.

$$\mathrm{Ext}^{s,k}\, v_n^{-1}BP_*/I_n \xrightarrow{v_n} \mathrm{Ext}^{s,k+2(p^n-1)}\, v_n^{-1}BP_*/I_n.$$

Even better, can define $M^n(i)$, which is periodic with respect to multiplication by $v_n^{p^{ni}}$. Then, can show $\mathrm{Ext}\, M^n = \varinjlim \mathrm{Ext}\, M^n(i)$; so $\mathrm{Ext}\, M^n$ is weakly periodic.

nth order phenomona in the ANSS: the subquotient of $\mathrm{Ext}\, BP_*$ given by $E_\infty^{n,*}$ of the CSS. We filter $\mathrm{Ext}\, BP_*$ by "order of periodicity," which is why its "chromatic."

Monochromatic Layers

Let's retell this story in topology: Set $N_0X = X$, $M_nX = L_nN_nX$, and define N_nX via cofiber sequences

$$N_nX \to M_nX \to N_{n+1}X.$$

Ravenel conjectured that for $X = S^0$, then BP homology recovers the chromatic resolution.

Theorem

Let C_nX be the fiber of $X \to L_nX$. Then, $N_nX = \Sigma^n C_{n-1}X$ and the fiber of $L_nX \to L_{n-1}X$ is $\Sigma^{-n}M_nX$. Thus, M_nX is the nth monochromatic component of X.

Splicing the above cofiber sequences, we get the *topological chromatic spectral sequence*

$$E_1^{s,t} = \pi_t M_s X \implies \pi_* L_\infty X.$$

Outline

Localization and Smashing

Conjecture (Smashing)

$E(n)$ localization is smashing: $L_n X \simeq L_{v_n^{-1}BP} X \simeq X \wedge L_n S^0$.

Conjecture (Localization)

For any spectrum X, we have $BP \wedge L_n X \simeq X \wedge L_n BP$.

In fact, Ravenel proved that $\pi_* N_n BP = N^n$ and $\pi_* M_n BP = M^n$, so the localization conjecture would imply that the topological CSS recovered the algebraic CSS.

Periodicity and Nilpotence

Considering our stratification, we can expect any p-local finite spectrum X to have a type n, i.e. the smallest n s.t. $K(n)_*X \not\simeq 0$.

Conjecture (Periodicity)

For X of type n, there exists a v_n-self map $f : \Sigma^k X \to X$ such that $K(n)_*(f)$ is given by multiplication by a p-th power of v_n, and $K(m)_*(f) = 0$ for $m \neq n$.

Conjecture (Nilpotence)

MU detects nilpotence. Equivalently,

1. *Any map $f : X \to \Sigma^{-k}X$ with $MU_*(f) = 0$ is nilpotent,*
2. *If $W \to X \to Y \xrightarrow{f} \Sigma W$ and $MU_*(f) = 0$, $\langle X \rangle = \langle W \rangle \vee \langle Y \rangle$.*

The Global Picture

Conjecture (Telescope)

If $f: X \to \Sigma^{-k}X$ is a v_n-self map, then $\langle f^{-1}X \rangle = \langle K(n) \rangle$.

As a consequence, we have that for X finite, there exist $K(n-1)_*$-acyclic spectra X_α s.t. $M_nX \simeq \operatorname{colim} M_nX_\alpha$, and for each X_α, there is an equivalence $M_nX_\alpha \to \Sigma^{2p^i(p^n-1)}M_nX_\alpha$.

This would give a geometric reason for the algebraic periodicity in the ANSS E_2 page!

Remark

For $n = 1$, all these conjectures are true; known to Adams. Devinatz-Hopkins-Smith proved all of these, except the telescope conjecture, and they all rest on the nilpotence theorem.

References

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