The Ravenel Conjectures and the Chromatic Decomposition

Rushil Mallarapu

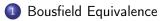
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Ravenel Conjectures

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BP and the Chromatic Zoo





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Bousfield Classes

Recall Bousfield localization at a homology theory E; we have a functor $L_E: \mathbf{SHC} \to \mathbf{SHC}$ and a universal natural E-equivalence $\eta: \text{ id } \Longrightarrow L_E$. How do we compare different localizations?

Definition

Two homology theories E, F are *Bousfield equivalent* if $E_*X = 0 \iff F_*X = 0$. Equivalence classes are denoted $\langle E \rangle$.

$$If E_*X = 0 \implies F_*X = 0, then we write \langle E \rangle \geq \langle F \rangle.$$

Solution Observe that
$$\langle 0 \rangle \leq \langle E \rangle \leq \langle S^0 \rangle$$
 for all *E*.

Some Properties

Theorem

 $\langle E \rangle = \langle F \rangle$ iff $L_E = L_F$. If $\langle E \rangle \le \langle F \rangle$, then $L_E L_F = L_E$ and there is a natural transformation $L_F \implies L_E$.

Theorem

For all $n \in \mathbb{Z}$, $\langle X \rangle = \langle \Sigma^n X \rangle$. If $X \to Y \to Z \to \Sigma X$ is a cofiber sequence, then $\langle X \rangle$, $\langle Y \rangle$, $\langle Z \rangle$ is \leq the wedge of the other two.

Some More Properties

Definition

Let R be a ring spectrum, and $\alpha \in \pi_*R$. This gives a self-map

$$\Sigma^{|lpha|} R = S^{|lpha|} \wedge R \xrightarrow{lpha \wedge 1} R \wedge R \xrightarrow{\mu} \mathsf{R},$$

which we'll denote α . Write R/α for the cofiber, and $\alpha^{-1}R$ for the colimit

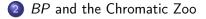
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$$\left[R \xrightarrow{\alpha} \Sigma^{-|\alpha|} R \xrightarrow{\alpha} \Sigma^{-2|\alpha|} R \xrightarrow{\alpha} \cdots \right]$$

Theorem

Given X and self-map $g: \Sigma^d X \to X$, then $\langle X \rangle = \langle \operatorname{colim} \Sigma^{-id} X \rangle \lor \langle Cg \rangle$. In particular, $\langle R \rangle = \langle \alpha^{-1} R \rangle \lor \langle R / \alpha \rangle$.

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A Little Motivation

Definition

Fix a prime p. The Brown-Peterson spectrum BP is a summand of MU localized at p with homotopy groups

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$$
 with $|v_n| = 2(p^n - 1)$.

Remark

In fact, BP_* has a similar relation to FGLs as MU_* ; it supports a universal *p*-typical formal group law!

Theorem (ANSS)

There is a spectral sequence

$$\Xi_2^{s,t} = \mathsf{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*X) \implies \pi_*L_{BP}X = \pi_*X_{(p)}.$$

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BP and Friends

Definition

For each $n \ge 0$, we have spectra given by

$$E(n) = v_n^{-1} BP/(v_{n+1}, v_{n+2}, \ldots),$$

$$K(n) = v_n^{-1} BP/(p, v_1, v_2, \dots, v_{n-1}, v_{n+1}, v_{n+2}, \dots).$$

In particular, $E(0) = K(0) = H\mathbb{Q}$, and E(1) is the Adams summand of p-local complex K-theory, $L_{(p)}K$.

Theorem

Let
$$I_n = (p, v_1, v_2, ..., v_{n-1})$$
. We have that
(1) $\langle v_n^{-1}BP/I_n \rangle = \langle K(n) \rangle$,
(2) $\langle v_n^{-1}BP \rangle = \langle E(n) \rangle$,
(3) $\langle E(n) \rangle = \bigvee_{i=0}^n \langle K(n) \rangle$.

Interpolating BP and S

BP sees everything about $SHC_{(p)}$. Before we go on, what does it miss?

Theorem

There exist Thom spectra X(n) for $n \ge 0$ satisfying

$$\left\langle S^{0}_{(p)} \right\rangle = \left\langle X(0) \right\rangle > \left\langle X(1) \right\rangle > \left\langle X(2) \right\rangle > \ldots > \left\langle BP \right\rangle$$

Sketch.

We have $\Omega SU \simeq BU$ by Bott Periodicity, and there is a natural map $SU(p^n) \rightarrow SU$, so we can define X(n) to be the (localized) Thom spectra of the bundle classified by $\Omega SU(p^n) \rightarrow BU$. Showing that $\langle X(n) \rangle > \langle X(n+1) \rangle$ is a computation with the Adams spectral sequence (globally, we interpolate using the James filtration). This map is a 2-fold loop map, so X(n) is a ring spectrum.

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The Chromatic Tower

Because $\langle E(n) \rangle \ge \langle E(n-1) \rangle$, we have natural transformations $L_n \to L_{n-1}$, so for any spectrum X, we get a tower

$$\cdots \to L_n X \to L_{n-1} X \to \cdots \to L_2 X \to L_1 X \to L_0 X = X_{\mathbb{Q}}.$$

Conjecture (Chromatic Convergence)

If X is a p-local finite spectrum, then $L_{\infty}X := \varprojlim L_nX$ is equivalent to X.

In fact, by the chromatic fracture square, the difference between L_n and L_{n-1} is measured by $L_{K(n)}$. So, we can reconstruct this tower from the "gluing data."

Connections with Algebra

Write $Ext(BP_*)$ for the E_2 page for $X = S^0$. Set $N^0 = BP_*$, $M^n = v_n^{-1}N^n$, and inductively define N^n by

$$0 \to N^n \to M^n \to N^{n+1} \to 0.$$

Splice these into the *chromatic resolution*:

$$0 \rightarrow BP_* \rightarrow M_0 = p^{-1}BP_* \rightarrow M_1 \rightarrow M_2 \rightarrow \ldots$$

The associated chromatic spectral sequence is

$$E_1^{s,t} = \operatorname{Ext}_{BP_*BP}^t (BP_*, M^s) \implies \operatorname{Ext}(BP_*).$$

Connections with Algebra

Why does this help? We can understand $\operatorname{Ext} v_n^{-1} BP_* / I_n$ via the *Morava* stabilizer group. It's v_n -periodic, i.e.

$$\operatorname{Ext}^{s,k} v_n^{-1} BP_* / I_n \xrightarrow{v_n} \operatorname{Ext}^{s,k+2(p^n-1)} v_n^{-1} BP_* / I_n.$$

Even better, can define $M^n(i)$, which is periodic with respect to multiplication by $v_n^{p^{ni}}$. Then, can show $\operatorname{Ext} M^n = \varinjlim \operatorname{Ext} M^n(i)$; so $\operatorname{Ext} M^n$ is weakly periodic.

nth order phenomona in the ANSS: the subquotient of Ext BP_* given by $E_{\infty}^{n,*}$ of the CSS. We filter Ext BP_* by "order of periodicity," which is why its "chromatic."

Monochromatic Layers

Let's retell this story in topology: Set $N_0X = X$, $M_nX = L_nN_nX$, and define N_nX via cofiber sequences

$$N_n X \to M_n X \to N_{n+1} X.$$

Ravenel conjectured that for $X = S^0$, then *BP* homology recovers the chromatic resolution.

Theorem

Let C_nX be the fiber of $X \to L_nX$. Then, $N_nX = \Sigma^n C_{n-1}X$ and the fiber of $L_nX \to L_{n-1}X$ is $\Sigma^{-n}M_nX$. Thus, M_nX is the nth monochromatic component of X.

Splicing the above cofiber sequences, we get the *topological chromatic spectral sequence*

$$E_1^{s,t} = \pi_t M_s X \implies \pi_* L_\infty X.$$

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Bousfield Equivalence

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Localization and Smashing

Conjecture (Smashing)

E(n) localization is smashing: $L_n X \simeq L_{v_n^{-1}BP} X \simeq X \wedge L_n S^0$.

Conjecture (Localization)

For any spectrum X, we have $BP \wedge L_n X \simeq X \wedge L_n BP$.

In fact, Ravenel proved that $\pi_*N_nBP = N^n$ and $\pi_*M_nBP = M^n$, so the localization conjecture would imply that the topological CSS recovered the algebraic CSS.

Periodicity and Nilpotence

Considering our stratification, we can expect any *p*-local finite spectrum X to have a type *n*, i.e. the smallest *n* s.t. $K(n)_*X \neq 0$.

Conjecture (Periodicity)

For X of type n, there exists a v_n -self map $f : \Sigma^k X \to X$ such that $K(n)_*(f)$ is given by multiplication by a p-th power of v_n , and $K(m)_*(f) = 0$ for $m \neq n$.

Conjecture (Nilpotence)

MU detects nilpotence. Equivalently,

• Any map $f: X \to \Sigma^{-k}X$ with $MU_*(f) = 0$ is nilpotent,

$$If W \to X \to Y \xrightarrow{f} \Sigma W \text{ and } MU_*(f) = 0, \ \langle X \rangle = \langle W \rangle \lor \langle Y \rangle.$$

The Global Picture

Conjecture (Telescope)

If $f: X \to \Sigma^{-k}X$ is a v_n -self map, then $\langle f^{-1}X \rangle = \langle K(n) \rangle$.

As a consequence, we have that for X finite, there exist $K(n-1)_*$ -acyclic spectra X_{α} s.t. $M_n X \simeq \operatorname{colim} M_n X_{\alpha}$, and for each X_{α} , there is an equivalence $M_n X_{\alpha} \to \Sigma^{2p^i(p^n-1)} M_n X_{\alpha}$.

This would give a geometric reason for the algebraic periodicity in the ANSS E_2 page!

Remark

For n = 1, all these conjectures are true; known to Adams. Devinatz-Hopkins-Smith proved all of these, except the telescope conjecture, and they all rest on the nilpotence theorem.

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