Formal Group Laws

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Outline

Introduction

In previous weeks we have discussed p-localizations of the stable homotopy category. This week, we'll reveal some additional structure on the p-local stable homotopy category, and discuss some connection to algebraic geometry.

Outline

Characteristic Classes

Definition

A **characteristic class** c is a natural assignment, to each complex vector bundle $E \to B$, an element $c(E) \in H^*(B; \mathbb{Z})$.

Theorem

Every characteristic class in a polynomial in the Chern classes $c_k(E) \in H^{2k}(B; \mathbb{Z})$.

Chern Classes

Definition

The **Chern classes** are defined by the following axioms. For the total Chern class $c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E)$:

1. $c(f^*E) = f^*c(E)$ for $f : X \to B$
2. $c(E \oplus E') = c(E)c(E')$
3. $c(\gamma) = 1 + c_1(\gamma) = 1 + t$, where $\gamma \to \mathbb{CP}^\infty$ is the tautological line bundle and $t \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$ is a generator.

Theorem

Let L and L' be line bundles. Then $c_1(L \otimes L') = c_1(L) + c_1(L')$.

Generalized cohomology

Does this work for a generalized cohomology theory E^*?

Definition

A **complex oriented cohomology theory** is a multiplicative cohomology theory E^* with an isomorphism $E^*(\mathbb{CP}^\infty) \cong E^*(\text{pt.})[[t]]$ for $t \in E^2(\text{pt.})$ a generator.

In this case, yes! In particular, since $\gamma \to \mathbb{CP}^\infty$ is universal, the Chern class $c_1(L)$ of any line bundle $L \to X$ is the pullback of t.

Generalized cohomology

What is the analogous theorem for tensor products of line bundles?

- We can compute $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E^*(\text{pt.})[[u, v]]$, where u and v are the pullbacks of t along the two projections $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$.
- Therefore the universal example of a tensor product, $\pi_1^*(\gamma) \otimes \pi_2^*(\gamma)$, has Chern class $F \in E^*(\text{pt.})[[u, v]]$.
- It follows that, for line bundles L and L' over B, $c_1(L \otimes L') = F(c_1(L), c_1(L'))$.

Outline

Formal Group Laws

Since the tensor product of line bundles is associative, commutative, and has a unit (the trivial bundle), F is a formal group law over $E^*(\text{pt.})$:

Definition

A (commutative, one dimensional) **formal group law** over a ring R is an element $F \in R[[x, y]]$ satisfying:

- $F(x, y) = F(y, x)$
- $F(x, 0) = F(0, x) = x$
- $F(x, F(y, z)) = F(F(x, y), z)$

Remark

These axioms imply that there is a unique inverse $i(x) \in R[[x]]$ such that $F(x, i(x)) = 0$.

Formal Group Laws: Motivation

Remark

A formal group laws can be thought of as the Taylor expansion of the group operation of a Lie group around the origin. In fact, the functor taking a Lie group to its Lie algebra factors through formal group laws.

Formal Group Laws: Homomorphisms

Definition

A homomorphism of formal group laws $F \to G$ is a power series h such that

$$h(F(x,y)) = G(h(x), h(y)).$$

Universal Formal Group Law

We can express a formal group law over R as

$$F = \sum_{i,j} a_{ij} x^i y^j,$$

where the $a_{ij} \in R$ satisfy some conditions:

- (Commutativity) $a_{ij} = a_{ji}$
- (Unital) $a_{10} = 1$; $a_{i0} = 0$ for $i > 1$
- Some complicated associativity relations

Therefore there is a universal formal group law over the ring $L = \mathbb{Z}[a_{ij}]/I$, where I is the ideal generated by these relations. Any formal group law over R is induced by a map $L \to R$.

Lazard's Theorem

Theorem (Lazard)

The **Lazard ring** $L = \mathbb{Z}[a_{ij}]/I$ *is isomorphic to* $\mathbb{Z}[t_1, t_2, \dots]$ *with* t_i *in degree* $2i$.

Remark

Recall that the cohomology ring for complex cobordism $MU^*(\text{pt.})$ is $\mathbb{Z}[t_1, t_2, \dots]$ with t_i in degree $2i$.

Quillen's Theorem

So L and $MU^*(\text{pt.})$ are abstractly isomorphism. But MU^* in fact a complex-oriented cohomology theory, so its formal group law is represented by a map $L \to MU^*(\text{pt.})$.

Theorem (Quillen)

The map $L \to MU^*(pt.)$ representing the formal group law associated with MU^* is an isomorphism.

n-series

Definition

The n-**series** $[n](t) \in R[[t]]$ of a formal group law F over R is defined inductively:

- $[0](t) = 0$
- For $n > 0$, $[n](t) = F([n-1](t), t)$
- For $n > 0$, $[-n](t) = i([n](t))$

Theorem

Each n-series is an endomorphism on F. The map $n \mapsto [n]$ is a homomorphism $\mathbb{Z} \to End(F)$.

Heights

From now on, we will fix a prime p.

Definition

Let F be a formal group law over a ring R with characteristic p. Let v_n be the coefficient of t^{p^n} in $[p](t)$.

- If $v_i = 0$ for $i < n$, F has height at least n.
- If F has height at least n and v_n is invertible in R, F has height exactly n.
- If F has height at least n for all n, we say F has height ∞.

Theorem

Height is invariant under isomorphism.

Heights

Theorem

- *The additive formal group law* $F(x, y) = x + y$ *has height* ∞.
- *The multiplicative formal group law* $F(x, y) = x + y + xy$ *has height 1.*
- *These two formal group laws are not isomorphic.*

Theorem

Let k *be an algebraically closed field of characteristic* p. *Two formal group laws* F *and* F' *over* k *are isomorphic if and only if they have the same height.*

Outline

Landweber Exact Functor Theorem

Suppose $MU^*(\text{pt.}) \to R$ is a map of rings representing a formal group law F. Is there a generalized cohomology theory E^* with associated formal group law F?
Idea: define the functor $E^*(-) = MU^*(-) \otimes_{MU^*(\text{pt.})} R$ (on finite complexes). This always satisfies all axioms except exactness.

Theorem (Landweber)

If, for each prime p, $(p, v_1, v_2, \dots)$ is a regular sequence in R, then E^* is a cohomology theory with associated formal group law F.

Morava K-theories

Definition

The Morava K-theories are a sequence of spectra $K(n)$.

- $K(0) = H\mathbb{Q}$
- $K(n)$ has $\pi_* K(n) = \mathbb{F}_p[v_n, v_n^{-1}]$ with v_n in degree $2(p^n - 1)$, and the associated formal group law has height n.

Morava K-theories

Let X be a spectrum of finite type.

Theorem (Ravenel)

If $K(n+1)_*(X) = 0$ *then* $K(n)_*(X) = 0$.

Definition

If n is the smallest value such that $K(n)_*(X)$ is nonzero, we say X is of type n.

$E(n)$-equivalence

Definition

A map is an $E(n)$-equivalence if it induces isomorphisms on $K(m)$ homology for all $m \leq n$.

Definition

The localization of a spectrum X at $E(n)$ equivalences is $L_n X = L_{K(0)\vee\cdots\vee K(n)}$.

Theorem

Localization at $E(n)$ is smashing: $L_n X \cong X \wedge L_n \mathbb{S}$.

Chromatic Fracture Square

Theorem

There is a pullback square

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X \end{array}$$

Slogan

$L_n\mathcal{SHC}$ is heights $\leq n$ in the chromatic filtration. $L_{K(n)}\mathcal{SHC}$ is height n. $L_{n-1}L_{K(n)}\mathcal{SHC}$ is the gluing data used to construct $L_n\mathcal{SHC}$ from $L_{n-1}\mathcal{SHC}$ and $L_{K(n)}\mathcal{SHC}$.

The End!

Thanks for listening!
Questions?