Steenrod Algebra and the Adams Spectral Sequence

Kenta Suzuki

Chroma Week 5

July 30, 2022

Table of Contents

Cohomology operations

Definition: A *stable cohomology operation* is a natural transformation of functors $\mathbf{Top}_* \to \mathbf{GrAb}$:

$$\theta \colon \widetilde{H}^*(-, \mathbb{F}_p) \to \widetilde{H}^*(-, \mathbb{F}_p),$$

where $\widetilde{H}^*$ is reduced singular cohomology.

By Brown representability and Yoneda's lemma, a cohomology operation

$$\widetilde{H}^*(-, \mathbb{F}_p) \cong [-, H\mathbb{F}_p]_{-*} \to [-, H\mathbb{F}_p]_{-*},$$

corresponds to an element of $[H\mathbb{F}_p, H\mathbb{F}_p]_{-*}$.

Definition: The mod-p *Steenrod Algebra*

$$\mathcal{A}^* := [H\mathbb{F}_p, H\mathbb{F}_p]_{-*}$$

is the $\mathbb{F}_p$-algebra of cohomology operations

$$\widetilde{H}^*(-, \mathbb{F}_p) \to \widetilde{H}^*(-, \mathbb{F}_p).$$

The algebra structure

- The Steenrod Algebra $\mathcal{A}^* = [H\mathbb{F}_p, H\mathbb{F}_p]_{-*}$ is a graded $\mathbb{F}_p$-algebra by addition and composition.
- For any spectrum Z, the cohomology $\widetilde{H}^*(Z; \mathbb{F}_p) := [Z, H\mathbb{F}_p]_{-*}$ is a $\mathcal{A}^*$-module, via the composition map

$$[Z, H\mathbb{F}_p]_{-*} \otimes [H\mathbb{F}_p, H\mathbb{F}_p]_{-*} \to [Z, H\mathbb{F}_p]_{-*}.$$

Henceforth, assume $p = 2$. In fact, $\mathcal{A}^*$ can be given the structure of a Hopf algebra!

Steenrod Squares

Theorem: *The Steenrod algebra $\mathcal{A}^*$ is generated by cohomology operations (which are group homomorphisms)*

$$Sq^n\colon \widetilde{H}^m(X, \mathbb{F}_2) \to \widetilde{H}^{m+n}(X, \mathbb{F}_2)$$

such that:

1. Sq^0 *is the identity*
2. *For* $x \in \widetilde{H}^n(X; \mathbb{F}_2)$, $Sq^n(x) = x^2$.
3. *For* $x \in \widetilde{H}^m(X; \mathbb{F}_2)$ *with* $m < n$, $Sq^n(x) = 0$.
4. *(Cartan formula)* $Sq^n(xy) = \sum_{i+j=n} Sq^i(x) Sq^j(y)$
5. Sq^1 *is the Bockstein, associated to the SES* $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$.
6. *(Adem relations) If* $0 < a < 2b$,

$$Sq^a Sq^b = \sum_{i=0}^{[a/2]} \binom{b-1-i}{a-2i} Sq^{a+b-i} Sq^i.$$

Equivalent forms of the Cartan formula

- The total square $Sq = Sq^0 + Sq^1 + \cdots : H^*(X) \to H^*(X)$ is a ring homomorphism.
- For $x \in \widetilde{H}^*(X; \mathbb{F}_2)$ and $y \in \widetilde{H}^*(Y; \mathbb{F}_2)$, the Künneth isomorphism gives

  $$x \otimes y \in \widetilde{H}^*(X; \mathbb{F}_2) \otimes \widetilde{H}^*(Y; \mathbb{F}_2) \cong \widetilde{H}^*(X \times Y; \mathbb{F}_2),$$

  then

  $$Sq^n(x \otimes y) = \sum_{i+j=n} Sq^i(x) \otimes Sq^j(y).$$

Example Calculation: $\mathbb{R}P^\infty$

Proposition:

$$\widetilde{H}^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[u], \text{ with } |u| = 1,$$

with the Steenrod algebra action given by:

$$Sq^j(u^{2^k}) = \begin{cases} u^{2^k} & j = 0 \\ u^{2^{k+1}} & j = 2^k \\ 0 & \text{other.} \end{cases}$$

Proof: Can run induction on $v_2(j)$, the 2-adic valuation. When $v_2(j) = 0$ (i.e., j is odd), we have

$$Sq^j(u^{2^k}) = \sum_{a+b=j} Sq^a(u^{2^{k-1}}) Sq^b(u^{2^{k-1}}) = 0.$$

Generators of $\mathcal{A}^*$

Definition: Let $\widetilde{\mathcal{A}^*}$ be the ideal of all the positive degree elements in $\mathcal{A}^*$. An element of $\mathcal{A}^*$ is *decomposable* if it is in the image of $\widetilde{\mathcal{A}^*} \otimes \widetilde{\mathcal{A}^*}$ under the multiplication map. Otherwise, an element is *indecomposable*.

Lemma: *The element Sq^i is indecomposable iff i is a power of 2.*

Proof: For $i = 2^k$ if Sq^i is decomposable, then there are $m_j \in \mathcal{A}^*$ with

$$Sq^i = \sum_{j=1}^{2^k-1} m_j \cdot Sq^j,$$

then in $\widetilde{H}^*(\mathbb{R}P^\infty, \mathbb{F}_2)$,

$$u^{2i} = Sq^i(u^i) = \sum m_j Sq^j(u^i) = 0,$$

a contradiction. The other direction is by Adem relations.
Corollary: *Sq^{2^k} generates $\mathcal{A}^*$ as an algebra.*

What is a Hopf algebra?

Definition: A *Hopf algebra* H over $\mathbb{F}_p$ has:

- a *multiplication* $\nabla\colon H \otimes H \to H$;
- a *comultiplication* $\Delta\colon H \to H \otimes H$;
- a *unit* $\eta\colon \mathbb{F}_p \to H$;
- a *counit* $\epsilon\colon H \to \mathbb{F}_p$; and
- an *antipode* $S\colon H \to H$,

making the appropriate diagrams commute.

Example: For a group G, the group ring $\mathbb{F}_p[G]$ is a Hopf algebra by:

- $\nabla\colon \mathbb{F}_p[G] \otimes \mathbb{F}_p[G] \to \mathbb{F}_p[G] : g \otimes h \mapsto gh$;
- $\Delta\colon \mathbb{F}_p[G] \to \mathbb{F}_p[G] \otimes \mathbb{F}_p[G] : g \mapsto g \otimes g$;
- $\eta\colon \mathbb{F}_p \to \mathbb{F}_p[G] : 1 \mapsto e_G$;
- $\epsilon\colon \mathbb{F}_p[G] \to \mathbb{F}_p : g \mapsto 1$; and
- $S\colon \mathbb{F}_p[G] \to \mathbb{F}_p[G] : g \mapsto g^{-1}$.

Hopf Algebra structure on $\mathcal{A}^*$ (cont.)

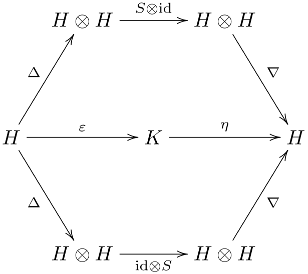
Theorem: *$\mathcal{A}^*$ is a Hopf algebra over $\mathbb{F}_2$ with comultiplication given by*

$$\Delta(Sq^k) = \sum_{i+j=k} Sq^i \otimes Sq^j$$

and antipode defined by

$$S(Sq^0) = Sq^0, \sum_{i+j=n} Sq^i \cup S(Sq^{n-i}) = 0 \text{ for } n \geq 1.$$

This is well-defined since $Sq^0 \in \mathcal{A}^*$ is the unit. It forces the commutativity of:



Dual of the Steenrod algebra

Definition: The *dual* of the Steenrod algebra $\mathcal{A}^*$ is defined by $\mathcal{A}_* := \mathrm{Hom}_{\mathbb{F}_p}(\mathcal{A}^*, \mathbb{F}_p)$.

Remark: The dual $\mathcal{A}_*$ also is a Hopf algebra. It is important since it has a simpler structure than $\mathcal{A}^*$:

Proposition For $p = 2$, we have $\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \dots]$, where $\xi_k = Sq^{2^{k-1}} \cdots Sq^2 Sq^1$.

Table of Contents

The spectral sequence

Notation: $\mathbb{Z}_p^\wedge = \lim \mathbb{Z}/p^n$ is the p-adic integers, and $\mathbb{Z}_{(p)}$ is the localization at the prime p. All cohomology is with $\mathbb{F}_p$-coefficients.

Theorem *For X and Y spectra of finite type with homotopy groups bounded below, and p a prime, there is a spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}^*}^{s,t}(H^*(Y), H^*(X)) \implies [X, Y_p^\wedge]_{t-s},$$

where $Y_p^\wedge$ denotes the p-completion of Y.

Example: If $X = \mathbb{S}$, then $[\mathbb{S}, Y_p^\wedge]_* = \pi_* Y \otimes \mathbb{Z}_p^\wedge$. (In particular, the case $Y = \mathbb{S}$ is important)

Theorem(dual) *For X and Y spectra of finite type with homotopy groups bounded below, and p a prime, there is a spectral sequence*

$$E^2_{s,t} = \mathrm{Ext}_{\mathcal{A}_*}^{s,t}(H_*(X), H_*(Y)) \implies [X, Y]_{t-s} \otimes \mathbb{Z}_{(p)}.$$

Some consequences

Remark: These spectral sequences only detect the p-part of the homotopy groups.

Example: Let $X = Y = \mathbb{S}$ and $p = 2$. Then, the first line of the spectral sequence can be calculated:

$$E_2^{1,t} = \mathrm{Ext}^{1,t}_{\mathcal{A}^*}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2\{h_i\} & \text{if } t = 2^i \\ 0 & \text{if } t \neq 2^i. \end{cases}$$

Proof Idea: Elements of $\mathrm{Ext}^{1,t}_{\mathcal{A}^*}(\mathbb{F}_2, \mathbb{F}_2)$ correspond to SES's

$$0 \to \mathbb{F}_2[t] \to M \to \mathbb{F}_2[0] \to 0,$$

which completely determines M as a graded $\mathbb{F}_2$-vector space. Whether or not there is a nontrivial $\mathcal{A}^*$-module structure depends on whether there is a indecomposable element which "intertwines" the two.

Which h_i's survive to the E_∞ page?

Fact: *The only permanent cycles are* h_0, h_1, h_2, h_3 *giving rise to elements in* $\pi_0(\mathbb{S}) \otimes \mathbb{Z}_2^\wedge, \pi_1(\mathbb{S}) \otimes \mathbb{Z}_2^\wedge, \pi_3(\mathbb{S}) \otimes \mathbb{Z}_2^\wedge$, *and* $\pi_7(\mathbb{S}) \otimes \mathbb{Z}_2^\wedge$. *(This is related to whether or not* S^n *is parallelisable.)*

Remark: The E_2-term $E_2^{s,t}$ can be calculated manually for small s and t, using explicit projective $\mathcal{A}^*$-module resolutions of $\mathbb{F}_p$.

Remark: Sometimes the "dual" spectral sequence is easier to calculate, since $\mathcal{A}_*$ has a much simpler description.

Table of Contents

Adams tower

Definition: An *Adams tower* for a spectrum Y is a diagram in $\mathcal{SHC}$

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow{\scriptstyle i_2} & & \\
 & & Y_2 & \longrightarrow & J_2 \\
 & & \downarrow{\scriptstyle i_1} & & \\
 & & Y_1 & \longrightarrow & J_1 \\
 & & \downarrow{\scriptstyle i_0} & & \\
Y & \xrightarrow{=} & Y_0 & \longrightarrow & J_0
\end{array}
$$

such that: (*continued*)

Adams tower (cont.)

- J_n is the homotopy cofiber of i_n
- $H^*(i_n) = 0$ for all n
- $H^*(J_n)$ is a projective $\mathcal{A}^*$-module for all n
- For all n and X,

$$[X, J_n]_* \xrightarrow{\simeq} \mathrm{Hom}^*_{\mathcal{A}^*}(H^*(J_n), H^*(X))$$

Fact: *For any spectrum Y, there is an Adams tower.*

Example (sphere)

Let $\overline{H\mathbb{F}_p}$ be the fiber

$$\overline{H\mathbb{F}_p} \to \mathbb{S} \xrightarrow{hur} H\mathbb{F}_p,$$

where $hur \in [\Sigma^\infty S^0, H\mathbb{F}_p] \cong \widetilde{H}^0(S^0; \mathbb{F}_p) = \mathbb{F}_p$ corresponds to the generator. The following is an Adams tower:

$$\begin{array}{ccc}
\overline{H\mathbb{F}_p}^{\wedge 2} & \longrightarrow & H\mathbb{F}_p \wedge \overline{H\mathbb{F}_p}^{\wedge 2} \\
\downarrow i_1 & & \\
\overline{H\mathbb{F}_p} & \longrightarrow & H\mathbb{F}_p \wedge \overline{H\mathbb{F}_p} \\
\downarrow i_0 & & \\
\mathbb{S} & \longrightarrow & H\mathbb{F}_p
\end{array}$$

The Construction

Let (Y_n, J_n) be an Adams tower for Y. Then, applying $[X, -]_*$ to the Adams tower gives an exact couple

$$
\begin{array}{ccc}
A^{**} = \bigoplus_{n\in\mathbb{Z}}[X, Y_n]_* & \xrightarrow{\quad i \quad} & \bigoplus_{n\in\mathbb{Z}}[X, Y_n]_* \, , \\
 {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\
 & B^{**} = \bigoplus_{n\in\mathbb{Z}}[X, J_n]_* &
\end{array}
$$

graded as $A^{s,t} = [X, \Sigma^s Y_s]_t$ and $B^{s,t} = [X, \Sigma^s J_s]_t$. Here:

- i is induced from $i_n \colon Y_{n+1} \to Y_n$ (degree $(-1, -1)$.)
- j is induced from $Y_n \to J_n$ (degree $(0, 0)$.)
- k is induced from $J_n \to \Sigma Y_{n+1}$ (degree $(1, 0)$.)

(*Remember:* $Y_{n+1} \to Y_n \to J_n$ is a distinguished triangle)

Exact couples

Definition: An *exact couple* is an exact sequence of abelian groups of the form

$$\begin{array}{ccc} A & \xrightarrow{\quad i \quad} & A \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & B & \end{array}.$$

Given an exact couple, $d := jk\colon B \to B$ is such that $d^2 = j(kj)k = 0$, so we can define the homology group $H(B) := \ker(d)/\mathrm{im}(d)$. The *derived* exact couple is:

$$\begin{array}{ccc} A' := i(A) & \xrightarrow{\quad i' \approx i \quad} & A' := i(A) \\ {\scriptstyle k' \approx k}\nwarrow & & \swarrow{\scriptstyle j' \approx j \circ i^{-1}} \\ & B' := H(B) & \end{array}.$$

Exact couples to spectral sequences

Let $A = A_0^{**}$ and $B = B_0^{**}$ be graded, and suppose the maps have degrees:

$$\begin{array}{ccc} A_0^{**} & \xrightarrow{i_0(-1,-1)} & A_0^{**} \\ & {\scriptstyle k_0(1,0)} \nwarrow \quad \swarrow {\scriptstyle j_0(0,0)} & \\ & B_0^{**}, & \end{array}$$

so the differential $j_0 k_0$ has degree $(1, 0)$. The derived couple has degrees:

$$\begin{array}{ccc} A_1^{**} & \xrightarrow{i_1(-1,-1)} & A_1^{**} \\ & {\scriptstyle k_1(1,0)} \nwarrow \quad \swarrow {\scriptstyle j_1(1,1)} & \\ & B_1^{**}, & \end{array}$$

so the differential $j_1 k_1$ has degree $(2, 1)$. Inductively, this gives rise to a spectral sequence with $E_n^{**} = B_n^{**}$ and $d_n = j_n k_n$, which has degree $(n + 1, n)$.

Construction (cont.)

Now for

$$A^{**} = \bigoplus_{s,t} [\Sigma^t X, \Sigma^s Y_s] \xrightarrow{\ i\ } A^{**}$$

$$A^{**} \xrightarrow{\ j\ } B^{**} = \bigoplus_{s,t} [\Sigma^t X, \Sigma^s J_s], \qquad B^{**} \xrightarrow{\ k\ } A^{**}$$

we have $E_1^{st} = [\Sigma^t X, \Sigma^s J_s]$, with differential induced by $\partial \colon J_n \to \Sigma J_{n+1}$. Here, I claim the sequence

$$Y \xrightarrow{\partial} J_0 \xrightarrow{\partial} \Sigma J_1 \xrightarrow{\partial} \cdots$$

gives a projective resolution of $\mathcal{A}^*$-modules

$$H^*(Y) \xleftarrow{\partial^*} H^*(J_0) \xleftarrow{\partial^*} H^{*+1}(J_1) \xleftarrow{\partial^*} \cdots .$$

Construction (cont., again)

Indeed, since $Y_{n+1} \xrightarrow{i_n} Y_n \to J_n$ is a cofibration, so gives rise to the LES

$$H^*(Y_{n+1}) \xleftarrow{i_n^*=0} H^i(Y_n) \leftarrow H^*(J_n) \leftarrow H^{*+1}(Y_{n+1}) \xleftarrow{i_n^*=0} H^{*+1}(Y_n),$$

i.e.,

$$0 \leftarrow H^*(Y_n) \leftarrow H^*(J_n) \leftarrow H^{*+1}(Y_n) \leftarrow 0.$$

Combining these exact sequences gives

$$0 \leftarrow H^*(Y) \leftarrow H^*(J_0) \leftarrow H^{*+1}(J_1) \leftarrow \cdots$$

Taking $\hom_{\mathcal{A}^*}(-, H^*(X))$ gives the projective resolution

$$\mathrm{Hom}_{\mathcal{A}^*}(H^*(J_0), H^*(X)) = [X, J_0]_* \xrightarrow{d} [X, J_1]* \xrightarrow{d} \cdots ,$$

so the homology of $E_1^{s,t}$ is

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}^*}^{s,t}(H^*(Y), H^*(X)).$$

References

- Bott-Tu, *Differential Forms in Algebraic Topology* (spectral sequences)
- Barnes-Roitzheim, *Foundations of Stable Homotopy Theory*