Bousfield Localization

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Outline

1. Conventions
2. Spanier-Whitehead Duality
3. Ring and Module Spectra
4. Bousfield Localization
5. p-Localization and p-Completion
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Outline

Conventions

- By Spec, we mean the category of CW spectra.

Note that any spectrum is weakly equivalent to a CW spectrum, so we don't lose much!

- By $\mathrm{Ho}(\mathrm{Spec})$, we mean the stable homotopy category of CW spectra.
- For A an abelian group, we denote the Eilenberg-MacLane spectrum on A to be HA.
- We denote the sphere spectrum $\Sigma^\infty S^0$ by $\mathbb{S}$.

Outline

Homology and Cohomology from Spectra

- Fix E a spectrum.
- Let X be a spectrum. In particular, we can take it to be an ordinary CW complex Y by $\Sigma^\infty Y$.

Definition

We define

$$E_* X := [\mathbb{S}, E \wedge X]_*,$$
$$E^* X := [X, E]_{-*}.$$

Theorem (Brown Representability)

Every cohomology theory on CW complexes is naturally isomorphic to E^* for some spectrum E.

There is a similar version for homology theories, satisfying the appropriately modified axioms.

Spanier-Whitehead Duality

- Fix a spectrum X.
- Then the functor $[X, -]_*$ is covariant and defines a homology theory.
- It follows that there is a spectrum $X^\vee$ which represents this functor: in other words, for all spectra Y, we have a natural isomorphism

$$[X, Y]_* \cong [\mathbb{S}, X^\vee \wedge Y]_*.$$

- Note that for $Y = \mathbb{S}$, we have $[X, \mathbb{S}]_* \cong [\mathbb{S}, X^\vee]_*$.
- The functor $X^\vee \wedge -$ is the "internal hom" which is right adjoint to $- \wedge X$.
- Warning: this dual only exists formally, but doesn't exhibit any nice properties in general. When X is a finite CW spectrum, then $X^\vee$ behaves as a bonafide dual.

Outline

Ring Spectra

Intuitively, a ring spectra is a spectra equipped with a "multiplication" operation and a unit.

Definition (Ring Spectrum)

A spectrum E is a **ring spectrum** equipped with maps $\mu : E \wedge E \to E$ and $\eta : \mathbb{S} \to E$ satisfying the associativity and unit axioms.

Both axioms are expressed in the usual commutative diagrams. See page 5 of Van Knoughnett's notes.

Module Spectra

Similarly, a (left) module spectrum over a ring spectrum E is just a spectrum equipped with a "(left) E-action."

Definition

A (left) **module spectrum** over a ring spectrum E is a spectrum F equipped with a map $\nu : E \wedge F \to F$ satisfying the usual associativity and unit diagrams.

The diagrams are the standard diagrams, and can be found on page 5 of Van Knoughnett's notes.

Examples

Example (Eilenberg-MacLane Spectra)

For any ring R, the Eilenberg-MacLane spectrum HR defines a ring spectrum. If M is an R-module, then HM is an HR-module spectrum.

Example ($\mathbb{S}$)

The sphere spectrum $\mathbb{S}$ is a ring spectrum, and since it is the unit, every spectrum is a module over $\mathbb{S}$.

Moore Spectrum

Definition

Let G be an abelian group. The **Moore spectrum** MG is constructed in the following way.

- Take a two-step free resolution (in Ab) of G:

$$0 \to \bigoplus_{I_2} \mathbb{Z} \xrightarrow{\rho} \bigoplus_{I_1} \mathbb{Z} \to G \to 0.$$

- Construct the following cofiber sequence, with $\pi_0(r) = \rho$:

$$\bigvee_{I_2} \mathbb{S} \xrightarrow{r} \bigvee_{I_1} \mathbb{S} \to MG.$$

Universal Coefficient Theorem

Theorem (Universal Coefficient Theorem)

Let E be a spectrum and G be an abelian group. Then let $EG = E \wedge MG$. There are the following natural short exact sequences (which do not split in general):

$$0 \to E_n X \otimes G \to (EG)_n X \to \mathrm{Tor}(E_{n-1} X, G) \to 0,$$
$$0 \to E^n X \otimes G \to (EG)^n X \to \mathrm{Tor}(E^{n-1} X, G) \to 0.$$

Proof of UCT

Proof.

Smashing the cofiber sequence of MG with $E \wedge - \wedge X$, we have

$$\bigvee_{I_2} E \wedge X \to \bigvee_{I_1} E \wedge X \to EG \wedge X.$$

Then taking graded maps from $\mathbb{S}$, we have the LES

$$\cdots \to \bigoplus_{I_2} E_n X \xrightarrow{f} \bigoplus_{I_1} E_n X \to (EG)_n(X)$$

$$\xrightarrow{g} \bigoplus_{I_2} E_{n-1} X \xrightarrow{h} \bigoplus_{I_1} E_{n-1} X \to \cdots ,$$

which we can split into short exact sequences: we have $\operatorname{coker} f = E_n X \otimes_{\mathbb{Z}} G$ and $\operatorname{im} g = \ker h = \operatorname{Tor}(E_{n-1}X, G)$. (Cohomology is proven the same way.) □

Outline

Overview

- We'll discuss Bousfield localization in general, and then specialize to the case of spectra.
- Broadly speaking, Bousfield localization is the analogue of localization of categories for model categories.
- Localization of categories is a formal procedure: given a category $\mathcal{C}$ and a class of morphisms $\mathcal{W}$ (under certain conditions), we want to "invert" the morphisms $\mathcal{W}$ to form a new category $\mathcal{C}[\mathcal{W}^{-1}]$, equipped with a functor $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$.
- This category should satisfy the following universal property: a functor $F : \mathcal{C} \to \mathcal{D}$ sends $\mathcal{W}$ to isomorphisms iff it factors through $\mathcal{C}[\mathcal{W}^{-1}]$.
- One example of localization of categories is the construction of the derived category by inverting quasi-isomorphisms.

Model Categories

Definition

Roughly speaking, a **model structure** on a category $\mathcal{C}$ consists of distinguished classes of morphisms called *weak equivalences*, *fibrations*, and *cofibrations*, satisfying certain axioms.

The details are not too important, so we'll leave it here as a general overview.

Local Objects

Let $\mathcal{C}$ be any category, and let $\mathbb{W}$ be a class of maps.

Definition

An object Y is $\mathbb{W}$-**local** if for all $f : A \to B$ in $\mathbb{W}$, then $f^* : \mathrm{Hom}_{\mathcal{C}}(B, Y) \to \mathrm{Hom}_{\mathcal{C}}(A, Y)$ is a bijection.

Think of $\mathbb{W}$-local objects as the objects "local to $\mathbb{W}$."

Definition

A map $f : A \to B$ is a $\mathbb{W}$-**equivalence** if for all $\mathbb{W}$-local objects Y, $f^* : \mathrm{Hom}_{\mathcal{C}}(B, Y) \to \mathrm{Hom}(A, Y)$ is a bijection.

We can think about $\mathbb{W}$-equivalences as "the maps which *should've* been in $\mathbb{W}$."
If we continue this procedure, we find it terminates after one step! Therefore, we can think about $\mathbb{W}$-equivalences as "completing $\mathbb{W}$."

Definition

A map $X \to Y$ is a $\mathbb{W}$-**localization** if it is a $\mathbb{W}$-equivalence and Y is $\mathbb{W}$-local. We then say that X **has a** $\mathbb{W}$-**localization**.

Definition

A *left Bousfield localization* of a model category $\mathcal{C}$ with a class of maps $\mathbb{W}$ is a model structure $L_{\mathbb{W}}\mathcal{C}$ on the underlying category $\mathcal{C}$, for which the cofibrations are the same and the weak equivalences as the $\mathbb{W}$-equivalences.

In general, the construction is quite technical, and is not known to always exist!
If all objects in $\mathcal{C}$ have $\mathbb{W}$-localizations, then there exists a functor $L_{\mathbb{W}} : \mathcal{C} \to \mathcal{C}$ sending X to a $\mathbb{W}$-localization. This functor essentially gives us a Bousfield localization of $\mathcal{C}$ by determining a reflective subcategory.

Bousfield Localization of Spec

Now let's apply this to (co)homology theories. Our hope is to treat those maps which induces isomorphisms on (co)homology as actual isomorphisms.
Since these theories are in bijection with spectra, first we fix a spectrum E.

Definition (E_*-equivalence)

A map $f : X \to Y$ of spectra is an E_*-**equivalence** if $E_* f$ is an isomorphism of graded abelian groups.

Our goal is to describe L_ESpec, the E-localization of Spec. This is just the Bousfield localization at $\mathcal{W}_E$, the class of E-equivalences.

Concrete Description of L_ESpec

Definition

A spectrum X is E_***-acyclic** if $E_*X = 0$. A spectrum Y is called E_***-local** if $[X, Y]_* = 0$ for every E_*-acyclic X.

Definition

An E_***-localization of (a spectrum)** X is an E_*-equivalence $X \to L_E X$ such that $L_E X$ is E_*-local.

We obtain a localization functor L_E, which is roughly a functorial choice of localizations. This L_E more or less contains the important information for localizing!

Rings and Modules

Let's see an example of something familiar.

Proposition

Let E be a ring spectrum and X a module spectrum over E. Then X is E_*-local.

Proof.

If A is E_*-acyclic, then $E_*A = 0 \implies E \wedge A \simeq *$. Therefore, for any $f : A \to X$, we can factor f as

$$A \xrightarrow{\sim} \mathbb{S} \wedge A \xrightarrow{\eta_E \wedge 1} E \wedge A \xrightarrow{1 \wedge f} E \wedge X \xrightarrow{\nu} X,$$

where ν is the module structure of X. But we can contract $E \wedge A$, so f is nullhomotopic, so $[A, X]_* = 0$. □

L_E Exists

Theorem (Existence of Homology Localizations of Spec)

For any E, L_E exists.

In this very important (and special) case, localization functors exist!
Describing L_E in general is too difficult, so we'll settle for describing what L_E looks like for Moore spectra.

Outline

Localization functors of Moore spectra

- Let G be an abelian group and MG the Moore spectrum associated to it.
- The localization functor is determined (up to weak equivalence) by the acyclic objects (as it is built from of these), so let's look at the MG-acyclic objects.
- From the Universal Coefficient Theorem, MG_*X is an extension of $\mathrm{Tor}(\pi_{n-1}(X), G)$ by $\pi_n(X) \otimes_{\mathbb{Z}} G$.
- Therefore, $MG_*X = 0$ iff both are zero.
- It is easy to see that this is completely determined by whether G is torsion or not, and the primes $p \in \mathbb{Z}$ for which G is uniquely p-divisible.
- This notion is known as **type of acyclicity**, which in turns classifies the localization functors L_{MG} associated to Moore spectra.

Type of Acyclicity

Definition

Let G_1, G_2 be abelian groups. Then they have the same **type of acyclicity** if:

1. G_1 is torsion iff G_2 is torsion, and
2. for every prime $p \in \mathbb{Z}$, G_1 is uniquely p-divisible iff G_2 is uniquely p-divisible.

Let $\mathcal{P}$ be a set of primes (possibly empty). It's an easy consequence that every abelian group has the same type of acyclicity as one of the following types of rings:

- $\mathbb{Z}_{\mathcal{P}} := \{a/b \mid p \nmid b \text{ for all } p \in \mathcal{P}\}$,
- $\mathbb{Z}/\mathcal{P} := \bigoplus_{p \in \mathcal{P}} \mathbb{Z}/p\mathbb{Z}$.

Therefore, we may consider only the above two groups in constructing the localization functor associated to the Moore spectrum.

$\mathcal{P}$-localization

Definition

When $G = \mathbb{Z}_{\mathcal{P}}$, we say L_{MG} is the **$\mathcal{P}$-localization** of X.

Theorem ($\mathcal{P}$-localization)

Let G be a localization of $\mathbb{Z}$. Then for any spectrum X, $L_{MG}(X) \simeq MG \wedge X$, and $\pi_* L_{MG}(X) = \pi_* X \otimes_{\mathbb{Z}} G$. The MG-local spectra are precisely the X for which $\pi_* X$ is uniquely p-divisible for all $p \notin \mathcal{P}$ (i.e., the p for which G is uniquely p-divisible).

The "unit" of this local category is the image of the sphere spectrum $\mathbb{S}$: namely, $MG \wedge \mathbb{S} \cong MG$. We call this the $\mathcal{P}$-**local sphere**.

$\mathcal{P}$-completion

Definition

When $G = \mathbb{Z}/\mathcal{P}$, we say L_{MG} is the $\mathcal{P}$-**completion** of X.

Theorem ($\mathcal{P}$-completion)

Let $G = \bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}$. Then for any spectrum X,

$$L_{MG}(X) \simeq \prod_{p\in\mathcal{P}} (\Omega M\mathbb{Z}_p)^\vee \wedge X.$$

If $\pi_* X$ is degreewise finitely generated, then

$$\pi_* L_{MG}(X) = \prod_{p\in\mathcal{P}} \pi_* X \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

For $G = \mathbb{Z}/p\mathbb{Z}$, there's a split short exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}_p, \pi_* X) \to \pi_* L_{M\mathbb{Z}/p\mathbb{Z}} X \to \mathrm{Hom}(\mathbb{Z}_p, \pi_{*-1} X) \to 0.$$

Outline

More Results

Recall that $EG = E \wedge MG$. Since we know how to localize at MG, we might ask how localization at EG is related to localization at E. It turns out that this depends only on E and the type of acyclicity of G.

Theorem (Bousfield)

Let E and X be spectra and G be an abelian group. If G is torsion or $E \wedge H\mathbb{Q} \not\simeq *$, then

$$L_{EG}X \simeq L_{MG}(L_E X).$$

Connective Spectra

Definition

X is a **connective spectrum** if $\pi_* X = 0$ for all $* < 0$.

Connective spectra are interesting because they are related to integer homology.

Proposition

If X is a connective spectrum, then X is $H\mathbb{Z}$-local. Furthermore, a map between connective spectra that induces an isomorphism on $H\mathbb{Z}$-homology (i.e., integer homology groups) is a stable equivalence.

We will also see that their localizations are easy to describe.

Localizing Connective Spectra

Another reason connective spectra are useful is that we essentially know everything about their localizations.

Theorem (Bousfield)

Let X and E be connective spectra and $G = \pi_0 E$. Then $L_E X \simeq L_{MG} X$.

We also know how to localize connective spectra at Eilenberg-MacLane spectra.

Theorem (Bousfield)

Let X be a connective spectrum and G an abelian group. Then $L_{HG} X = L_{MG} X$.

There are also results relating describing localization *of* Eilenberg-MacLane spectra, but it is more complicated.

The End $\ddot\smile$

Thanks for listening!!

Questions?