Bundles, Bordism, and Thom Spectra

Jonathan Buchanan

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Table of Contents

Summary

Last Week's Talk

Last week, we were given an introduction to the stable homotopy category and some examples of spectra.

This Week's Talk

This week, we will introduce fiber bundles, principal G-bundles, classifying spaces, and vector bundles. Then, we will discuss bordism and the Thom spectra, as well as computation results.

Table of Contents

Fiber Bundles

A fiber bundle makes precise the notion of a "local product". A **fiber bundle** with **fiber** F is a surjective map $p : E \to B$, where B has an open cover $\mathcal{U}$ along with homeomorphisms $\phi_U : p^{-1}(U) \to U \times F$ such that the diagrams

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi_U} & U \times F \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle \pi_1} & \\ U & & \end{array}$$

commute. The open sets $U \in \mathcal{U}$ are called **trivializing neighborhoods**, and the maps ϕ_U are **local trivializations**.

Examples of Fiber Bundles

- Covering spaces over a path-connected space are fiber bundles with discrete fibers.
- The Möbius Band is a non-trivial fiber bundle over S^1 with fiber $\mathbb{R}$.
- For any smooth n-manifold M, the tangent bundle TM is a fiber bundle with fiber $\mathbb{R}^n$..

Group Actions

Let G be a topological group. A **left G-space** is a space X and a continuous map $G \times X \to X$ that is a group action of G on X as a set. We can make a category of left G-spaces, where the morphisms are **G-equivariant maps**. That is, maps $f : X \to Y$ such that $f(gx) = gf(x)$.

Principal Bundles

If G is a topological group, a **principal G-bundle** over B is an equivariant map $p : E \to B$ of right G-spaces, where B has the trivial action, admitting an open cover $\mathcal{U}$ of B with G-homeomorphisms $\phi_U : p^{-1}(U) \to U \times G$ such that the diagrams

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi_U} & U \times G \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle \pi_1} & \\ U & & \end{array}$$

commute.

Categorical Aspects of Principal Bundles

The principal G-bundles over a space B form a category, where the morphisms between $p : E \to B$ and $p' : E' \to B$ are equivariant maps $f : E \to E'$ such that

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E' \\ {\scriptstyle p}\downarrow & \swarrow {\scriptstyle p'} & \\ B & & \end{array}$$

The category of principal G-bundles over B is a groupoid. We will write the set of isomorphism classes of principal G-bundles over B as $\mathrm{Bun}_G B$.

Pullback Bundles

Given a principal G-bundle $p : E \to B$ and a continuous map $f : A \to B$, we can form the **pullback bundle** $f^*p : f^*E \to A$, which is the pullback in the category of spaces

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \Big\downarrow{\scriptstyle f^*p} & & \Big\downarrow{\scriptstyle p} \\ A & \xrightarrow{\ f\ } & B \end{array}$$

This makes Bun_G into a contravariant functor from the category of spaces to the category of sets.

Table of Contents

Presheaves and Representable Functors

- If $\mathcal{C}$ is category, a **presheaf** on $\mathcal{C}$ is a functor $\mathcal{C} \to \mathrm{Set}$. The presheaves on $\mathcal{C}$ form a category $\mathrm{Pre}\mathcal{C}$, where the morphisms are natural transformations.
- If $\mathcal{C}$ is a locally small category, there is a functor $\mathbf{y} : \mathcal{C} \to \mathrm{Pre}\mathcal{C}$ called the **Yoneda embedding** taking $X \in \mathcal{C}$ to the presheaf $Y \mapsto \mathcal{C}(Y, X)$.
- A presheaf F is said to be **representable** if it is isomorphic to $\mathbf{y}X$ for some $X \in \mathcal{C}$.

The Yoneda Lemma

Theorem

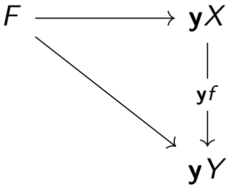
Let F be a presheaf on $\mathcal{C}$. Then for any $X \in \mathcal{C}$, there is a bijection between natural transformations $\mathbf{y}X \to F$ and the set $F(X)$. This is natural in both F and X.

The basic idea is that given a natural transformation $T : \mathbf{y}X \to F$, we get an element of $F(X)$ by evaluating $T_X : \mathcal{C}(X, X) \to F(X)$ at id_X. Given an element $x \in F(X)$, we may define a natural transformation $T : \mathbf{y}X \to F$ by taking a map $f \in \mathcal{C}(Y, X)$ and then setting $T_Y(f) = (F(f))(x)$. It is easy to check that everything here is natural and these mappings are inverses.

Consequences of The Yoneda Lemma

Corollary

If F is a representable presheaf, then for any two objects $X, Y \in \mathcal{C}$ both representing F, there is a unique isomorphism $f : X \to Y$ such that the diagram



commutes.

Table of Contents

Characterization of Classifying Spaces

To work with bundles in homotopy theory, we will need to show that they have nice properties under homotopy. Fortunately, they do!

Theorem

The functor Bun_G is homotopy invariant.

We can therefore regard Bun_G as a functor $\mathrm{hTop}^{\mathrm{op}} \to \mathrm{Set}$, and we can ask for even more: representability. A representing object for Bun_G will be called a **classifying space**. A principal G-bundle $E \to B$ is **universal** if E is contractible.

Theorem

A space BG is a classifying space if and only if it supports a universal bundle.

Existence and Uniqueness of Classifying Spaces

From the Yoneda lemma, it follows that classifying spaces are unique for the purposes of homotopy theory:

Corollary

Any two classifying spaces for G are homotopy equivalent.

There are several ways to construct classifying spaces:

- The Milnor Join (a join of countably many copies of G).
- The infinite Grassmannian for classical Lie groups.
- The geometric realization of the nerve of G regarded as a category.

The main takeway is this:

Theorem

There is a functor B from topological groups to topological spaces sending each group G to a classifying space BG.

Vector Bundles

Now that we've examined the general theory, we can discuss **vector bundles**, which are fiber bundles with fiber $\mathbb{R}^n$ (or $\mathbb{C}^n$) and a linear structure on each fiber compatible with the trivialization. A map of vector bundles is a continuous map of bundles over B that is linear in each fiber. However, by assigning a continuously varying inner product to a vector bundle, we can identity the orthonormal bases of each fiber to obtain natural isomorphisms $\mathrm{Vect}_n \to \mathrm{Bun}_{O(n)}$, where $\mathrm{Vect}_n : \mathrm{hTop}^{\mathrm{op}} \to \mathrm{Set}$ is the functor taking each space to its set of isomorphism classes of n-dimensional vector bundles.

A Warning

As usual, we must make concessions somewhere to avoid pathologies in point-set topology:

Numerable Bundles

Everything we have discussed above is true for **numerable fiber bundles**, which are bundles with a trivializing cover admitting a partition of unity.

In the case of nice spaces (paracompact Hausdorff spaces, to be precise, which include CW complexes), this restriction ends up not mattering. This approach has the advantage that we can work with all spaces and only consider nice bundles over them.

Table of Contents

Bordism

Two smooth closed n-manifolds without boundary M and N are **bordant** if there is an $(n+1)$-manifold W and a diffeomorphism $M \coprod N \to \partial W$. Bordism is an equivalence relation, and we denote the set of bordism classes of n-manifolds by Ω_n^O. If we consider the empty set as an n-manifold for all natural numbers n, then Ω_n^O has an abelian group structure under disjoint union with identity $\emptyset$. Furthermore, Ω^O, the union of all bordism classes, has a graded commutative ring structure with identity element $\{*\}$, where multiplication is given by the product of smooth manifolds.

Thom Spaces

For a vector bundle $p : E \to B$, the **Thom Space** $\mathrm{Th}p$ is defined as the quotient $D(p)/S(p)$, where $D(p)$ is the unit disk bundle of p and $S(p)$ is the unit sphere bundle (in some metric).

Proposition

If B is compact, $\mathrm{Th}p$ is homeomorphic to the one-point compactification of E.

Proposition

If $p : D \to A$ and $q : E \to B$ are two vector bundles, then letting $p \times q : D \times E \to A \times B$ be the product bundle, there is a homeomorphism

$$\mathrm{Th}(p \times q) \cong \mathrm{Th}p \wedge \mathrm{Th}q.$$

The Thom Isomorphism

Theorem

Let R be a commutative ring an $p : E \to B$ an R-oriented n-plane bundle. Then there is an isomorphism $\tilde{H}^{k+n}(\mathrm{Th}p; R) \to H^k(B; R)$.

This isomorphism is given by $x \mapsto i^*(x)\,u$, where $u \in H^n(\mathrm{Th}p; R)$ is a unique class called the **Thom class** and $i : B \to \mathrm{Th}p$ is the inclusion of B as the zero section.

Thom Spectra

Let $\xi_n : EO(n) \to BO(n)$ be the universal bundle. The **Thom spectrum** MO is the sequence of spaces $MO_n = \mathrm{Th}\xi_n$ and the structure maps $\alpha_n : \Sigma MO(n) \to MO(n+1)$ are the composition

$$\Sigma MO(n) \cong \mathrm{Th}\xi_n \wedge S^1 \cong \mathrm{Th}\xi_n \oplus \mathbb{R} \to \mathrm{Th}\xi_{n+1},$$

where the map $\mathrm{Th}\xi_n \oplus \mathbb{R} \to \mathrm{Th}\xi_{n+1}$ is induced by the classifying map of $\xi_n \oplus \mathbb{R}$. This spectrum is an associative and commutative ring spectrum. This construction generalizes to groups that embed in $O(n)$, so we also have spectra MSO and MU, for example.

Homotopy Groups of Thom Spectra

Theorem

There is an isomorphism $\Omega_n^O \to \pi_n MO$.

To construct this homomorphism, suppose we are given a smooth closed n-manifold M. We embed it in $\mathbb{R}^{n+k}$ and form the normal bundle $\nu : E \to M$ over M for this embedding. Using a tubular neighborhood, we can think of E as being embedded in an open set of $\mathbb{R}^{n+k}$. The Thom space construction then gives a map of one-point compactifications $S^{n+k} \cong \mathbb{R}_+^{n+k} \to E_+ \cong \mathrm{Th}\nu$. Then using the classifying map of the normal bundle, we get a map $\mathrm{Th}\nu \to \mathrm{Th}\xi_k$. This determines an element of $\pi_n MO$. Checking that this is a homomorphism of groups and has an inverse is much more involved.

Generalized Bordism

The spectra MSO and MU have a nice geometric interpretation in terms of bordism as well. The spectrum MSO represents **oriented bordism**, where we consider orientation-preserving diffeomorphisms from the direct sum of two oriented manifolds to the boundary of an oriented manifold. The spectrum MU represents **complex bordism**, where the bordisms preserve complex vector space structures extending the real vector space structure on the stable tangent bundle.

Calculation of the Bordism Ring

Using the Adams spectral sequence and the Steenrod algebra we can calculate the homotopy groups of various generalized Thom spectra. We also know some of the manifold representatives, but not all (this is the topic of my research)!

Theorem

The ring Ω^O is isomorphic to the polynomial ring over $\mathbb{F}_2$ with a generator for each dimension n not equal to $2^k - 1$.

Theorem

The ring Ω^U is isomorphic to the polynomial ring over $\mathbb{Z}$ with a generator for each even dimension.