# Introduction to this Summer's Topics!

## Preliminaries, Stable Homotopy Category, and the Nilpotence Theorem

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# Introduction

| | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ | $n=7$ | $n=8$ | $n=9$ | $n=10$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_n(S^n)$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $\pi_{n+1}(S^n)$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\pi_{n+2}(S^n)$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\pi_{n+3}(S^n)$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ | $\mathbb{Z} \oplus \mathbb{Z}_{12}$ | $\mathbb{Z}_{24}$ | $\mathbb{Z}_{24}$ | $\mathbb{Z}_{24}$ | $\mathbb{Z}_{24}$ | $\mathbb{Z}_{24}$ | $\mathbb{Z}_{24}$ |
| $\pi_{n+4}(S^n)$ | 0 | $\mathbb{Z}_{12}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | 0 | 0 | 0 | 0 |
| $\pi_{n+5}(S^n)$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | 0 |
| $\pi_{n+6}(S^n)$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_{24} \oplus \mathbb{Z}_3$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\pi_{n+7}(S^n)$ | 0 | $\mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_{30}$ | $\mathbb{Z}_{60}$ | $\mathbb{Z}_{120}$ | $\mathbb{Z} \oplus \mathbb{Z}_{120}$ | $\mathbb{Z}_{240}$ | $\mathbb{Z}_{240}$ |

# Important Definitions

- Suspension: given $X$ a space, we define $\Sigma X$ as

$$X \times I / X \times \{0\}, X \times \{1\}$$

  This gives us an endofunctor on Top.
- Reduced suspension: for a base point $x_0 \in X$, we make the further identification $(x_0, t_1) \sim (x_0, t_2)$ for all $t_1, t_2$. For nice enough spaces (eg CW complexes), reduced suspension and ordinary suspension are homotopy equivalent, so we often use the notation $\Sigma$ interchangeably.
- Loop space: given a (pointed) $X$, $\Omega X$ is the space of (pointed) maps $S^1 \to X$.
- If $\Sigma^i f$ is nullhomotopic for some $i$, $f$ is *stably nullhomotopic*; otherwise it's *stably essential*.

# Important Definitions Continued

- Smash product: for spaces $X, Y$ with base points $x_0, y_0$, we define $X \wedge Y$ to be

$$X \times Y/(x_0, y) \sim (x, y_0)$$

for all $x, y$. Reduced suspension is a special case of the smash product: $\Sigma X \simeq X \wedge S^1$.

- The smash product makes the category of pointed spaces into a *symmetric monoidal category*. For example, check that the unit object is $S^0$.
- A *homology theory* $E_*$ is a homotopy invariant functor from Top to the category of graded abelian groups (satisfying Eilenberg-Steenrod). The natural map

$$\epsilon : E_*(X) \to E_*(\text{pt})$$

is called the *augmentation map* and its kernel is the *reduced homology* of $X$.

# Some Motivation

**Remark:** for many homotopy invariant functors - such as homology - nullhomotopic maps are sent to trivial maps. But, the converse is rarely true.

**Problem:** find such a (hopefully easily computable) functor such that $F(f)$ is trivial iff $f$ is nullhomotopic.

This problem is probably impossible. But, restricting to nicer spaces makes it possible to give special cases in which the converse is true. The *Nilpotence Theorem* is such a result.

**Problem:** find a reduced homology theory (preferably one easy to compute) such that $\overline{E_*}(f)$ is trivial iff $f$ is stably nullhomotopic.

The *generating hypothesis* (Freyd) says that stable homotopy is such a homology theory.

# Freudenthal Suspension Theorem

**Theorem:** Let $X$ be an $n$-connected CW complex. Then

$$\pi_i(X) \simeq \pi_{i+1}(\Sigma X)$$

for $i \leq 2n$. In particular, for $n > i + 1$, we have

$$\pi_{i+n}(S^n) \simeq \pi_{i+n+1}(\Sigma S^n) \simeq \pi_{i+n+1}(S^{n+1})$$

ie if you if you continue to take suspensions of $S^n$, eventually the $i$th homotopy group 'stabilizes' and stays the same. The stable homotopy groups of spheres motivate the *stable homotopy category.*

We can see that $\pi_*^S$ is really a graded ring; if $\alpha \in \pi_m^S$ and $\beta \in \pi_n^S$ are represented by maps $f, g$ into $S^n$, then $\alpha\beta$ is represented by

$$f \circ \Sigma^m g : S^{n+m+k} \to S^{n+k} \to S^k$$

# Nilpotence Theorem I

**Definition:** a map $\Sigma^d X \to X$ for some $d$ is a *self-map* of $X$. $f$ is *nilpotent* if one of its suspensions is nullhomotopic; otherwise it is *periodic*.

**Nilpotence Theorem (Statement 1):** there exists a homology theory $MU_*$ such that a self map $f$ of a finite CW complex is stably nilpotent iff some iterate of $\overline{MU_*}(f)$ is trivial.

(We will see two stronger forms of the theorem soon).

# Brown's Representability Theorem

Gives conditions for a contravariant functor $F$ from the homotopy category of pointed CW complexes to Sets to be representable. These conditions are:

1. $F(\vee_\alpha X_\alpha) \simeq \prod_\alpha F(X_\alpha)$
2. $F$ sends homotopy pushouts to weak pullbacks (Mayer-Vietoris Axiom)

Then, $F$ is isomorphic to $\mathrm{Hom}(-, X)$ for some $X$ in a natural way.

For example, taking $F$ to be $H^i(X; A)$ for an abelian $A$, we see that the conditions are fulfilled, so there is a representing space $X$. This space is $K(A, i)$ (gives a way of proving the existence of Eilenberg-Maclane spaces).

# Spectra and the Stable Homotopy Category

# Intuitive Idea/Motivation?

We know that $\Omega$ and $\Sigma$ are *adjoints* of each other. That is, there is a natural bijection

$$[\Sigma X, Y] \simeq [X, \Omega Y]$$

(Eckmann-Hilton duality). Furthermore, we know that there is a natural group structure on the two sets above.

However, $\Sigma$ and $\Omega$ are not equivalences of categories. For example, consider the Hopf Fibration $S^3 \to S^2$.

Thus (as I understand it so far) the *stable homotopy category* is a way of working in a category in which there are analogues to $\Sigma$ and $\Omega$ such that every object is actually isomorphic to a suspension or loop object.

# Axiomatic Definition

Instead of going through the details of constructing this category ground-up, we save time by describing the properties of the stable homotopy category and then just asserting that such a category exists. We claim that our category satisfies the following:

- There is a functor $\Sigma^\infty : \mathrm{HoTop} \to \mathrm{HoSpectra}$.
- There is a suspension functor $\Sigma$ agreeing with the usual suspension functor, ie the following diagram commutes:

$$\begin{array}{ccc} \mathrm{HoTop} & \xrightarrow{\Sigma} & \mathrm{HoTop} \\ \big\downarrow{\scriptstyle \Sigma^\infty} & & \big\downarrow{\scriptstyle \Sigma^\infty} \\ \mathrm{HoSpectra} & \xrightarrow{\Sigma} & \mathrm{HoSpectra} \end{array}$$

- There is similarly a functor $\Omega^\infty$, right adjoint to $\Sigma^\infty$, and a corresponding loop functor $\Omega : \mathrm{HoSpectra} \to \mathrm{HoSpectra}$.
- Both $\Sigma$ and $\Omega$ are equivalences of categories from HoSpectra to itself, and also form an equivalence of categories together.

# Axiomatic Definition Continued

- There is a natural way to turn $[X, Y]$ into an abelian group. $\Sigma$ and $\Omega$, as usual, form an adjunction too.
- Our category has wedge sums $X \vee Y$ and products $X \times Y$. Additionally, it has a zero object, coming from $*$ in Top. The natural maps

$$X \vee * \to X, X \to X \times *, X \vee Y \to X \times Y$$

  are all isomorphisms (the third map is not usually an isomorphism for normal topological spaces!)
- If $X$ is a retract of $A$, it contains $A$ as a summand.
- We can view $[X, Y]$ as a *graded* abelian ring by taking its $i$th piece to be $[\Sigma^i X, Y]$.

(ie, we also know that our category is an *additive* category, though it is not abelian).

# Properties of the Stable Homotopy Category

- Define the *sphere spectrum* to be $\mathbb{S} = \Sigma^\infty S^0$. We can define the stable homotopy groups of an object $X$ in this category to be

$$\pi_n(X) = [\Sigma^n \mathbb{S}, S]$$

  When $X$ is a sphere, these groups are naturally isomorphic to the usual stable homotopy groups as discussed before.
- Note that $n$ can actually be any integer, since $\Sigma$ has a genuine inverse equivalence $\Omega$ now.
- In general, objects with trivial negative stable homotopy groups are called *connected spectra*. All objects coming from spaces are connected spectra.
- Whitehead's Theorem: if $f : X \to Y$ induces an isomorphism on the stable homotopy groups, then $f$ itself is an isomorphism (stable homotopy detects isomorphisms).

# Further Properties

- We can define a smash product $X \wedge Y$, turning HoSpectra into a symmetric monoidal category as with spaces (this is analogous to the tensor product of abelian groups). The unit object is $\mathbb{S}$.
- $\Sigma^\infty$ respects the smash product (ie it is monoidal).
- The objects of HoSpec define (co)homology theories on CW. It turns out, by Brown representability, that every cohomology theory on finite CW complexes is represented by an object in HoSpectra, unique up to canonical isomorphism.

# Distinguished Triangles

We have classical cofiber and fiber sequences:

$$A \to X \to X/A \to \Sigma A$$

and

$$\Omega B \to F \to E \to B$$

We generalize these sequences to *distinguished triangles* in HoSpectra: these are defined as tuples $(X, Y, Z, f, g, h)$ fitting into a diagram

$$X \to Y \to Z \to \Sigma X$$

We get a long exact sequence of homotopy groups from distinguished triangles, and $\Sigma^\infty, \Omega^\infty$ both send cofiber sequences to distinguished triangles and vice versa.

Many of the other properties of classical (co)fiber sequences also apply; for example, all maps have (homotopy) cofibers. Also note that cofiber and fiber sequences coincide - another reason why this category is 'stable.'

# Ok, but what IS this category?

There are several different definitions of spectra. I find sequential definitions easiest to think about. Adams (1974) defined a spectrum as a sequence $\{E_n\}$ of CW complexes together with inclusions $\Sigma E_n \to E_{n+1}$ which are cellular maps.

**Examples:**

1. An Eilenberg-Maclane spectrum has $n$th space $K(A, n)$ for a given abelian group $A$.
2. The sphere spectrum has $n$th space $S^n$, and the maps are the canonical homeomorphisms $\Sigma S^n \to S^{n+1}$. (It is a *ring spectrum*).
3. The suspension spectrum of a space $X$ in general is given recursively, by $X_n = S^n \wedge X$. We denote this spectrum by $\Sigma^\infty X$.

**Homotopy groups:** for a spectrum $E$, we define $\pi_n(E)$ to be the limit over $k$ of $\pi_{n+k}(E_k)$ (where the maps $\pi_{n+k}(E_k) \to \pi_{n+k+1}(E_{k+1})$ are the maps given by the composition $E_k \to \Sigma E_k \to E_{k+1}$).

# More on the Nilpotence Theorem

# Different Forms of the Nilpotence Theorem

Take for granted (for now): $MU$ is a spectrum representing a certain cohomology theory, $MU_*$. It has the following very nice properties.

1. Self-map form: if $X$ is a finite spectrum and $\alpha : \Sigma^d X \to X$ is a self-map, then $MU_*(\alpha) = 0$ implies that $\alpha$ is nilpotent.
2. Smash product form: if $X$ is a finite spectrum, $f : X \to Y$ is a map of spectra, then if $\mathrm{id}_{MU} \wedge f$ is nullhomotopic, $f$ is smash nilpotent. (ie $f^{\wedge k} : X^{\wedge k} \to Y^{\wedge k}$).
3. Ring spectrum form: let $R$ be a ring spectrum and $h : \pi_*(R) \to MU_*(R)$ be the Hurewicz homomorphism. Every element of the kernel of $h$ is nilpotent. (Weak ring spectrum form is just this statement, but for connected finite spectra).

# Applications

**Nishida's Theorem:** every element of positive degree in $\pi_*^S$ is nilpotent.

Proof: Serre finiteness tells us that all $\pi_d^S$ are finite abelian, so all elements are torsion. On the other hand, a result of Novikov tells us that $\pi_*(MU) \simeq \mathbb{Z}[x_1, x_2 ...]$ with $|x_i| = 2i$ is torsion free. Thus all elements of $\pi_d^S$ are in the kernel of $\pi_d^S \to \pi_*(MU)$. By the ring spectrum form, we are done.

# Thank you!/Questions?