A WALK THROUGH THE WEIL CONJECTURES

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Contents

Étale cohomology is a ubiquitous and useful tool in studying the topology of algebraic varieties, and has been successfully employed in proving the Weil conjectures, deep-reaching statements about the arithmetic content of such varieties over finite fields. We give an overview of étale cohomology, introduce the (now proved) Weil conjectures, and discuss a much simpler proof in the case of curves.

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1. Étale Morphisms and Étale Cohomology

The notion of *étale* is pervasive in algebraic geometry. It provides a notion of "local isomorphism" that is much closer to the intuitions of analytic varieties; in particular, the étale topology on a scheme is fine enough for a useful version of cohomology groups, enabling one to do "algebraic topology" with varieties. I learned this material from [Tam94] and [FK88], but found [Mil13] surprisingly readable.

Étale morphisms and the étale topology. To start, we review the definition and intuitions behind étale morphisms.

Definition 1.1. Let $f\colon X \to Y$ be a locally finitely presented morphism of schemes. We say f is *étale* if f is flat and unramified; that is, for each $y \in Y$, the fiber X_y as a $\kappa(y)$-scheme is the union of the spectrum of finite separable extensions of $\kappa(y)$ [Tam94, 1.1.1].

Étale morphisms are quite nice: they are stable under composition, base change, and cancellation; i.e. if X, Y are schemes étale over some base S and $f\colon X \to Y$ is a map of S-schemes, then f is automatically étale. Open immersions are also étale; this translates into the étale topology being finer than the Zariski one.

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Gaining a good intuition for what "étale" means took me a while: here are some alternative definitions/perspectives I like. First, let $f\colon X \to Y$ be a morphism and $T \nrightarrow T'$ a first order thickening (i.e. $T \subset T'$ is a closed subscheme defined by a square-zero sheaf of ideals). Then, we can consider lifts of T-rational points of X over Y to T'-rational points, i.e. dotted arrows in the diagram

$$\begin{array}{ccc} T & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ T' & \longrightarrow & Y \end{array}$$

Then, we say f is *formally unramified* if there exists at most one lift. It is *formally smooth* if there exists at least one lift, and and is *formally étale* if there is exactly one lift [Sta22, 02HG]. If we add lfp to this, we can drop the "formal," and we see that étale is the same as smooth and unramified.

There's also a differential criterion for these properties, more analogous to the differential-geometric notions of immersions, submersions, and local isomorphisms. It turns out that a morphism f is formally unramified iff $\Omega_{X/Y} = 0$, and f is smooth iff $\Omega_{X/Y}$ is finite locally free (of constant rank). Thus, being unramified is like being an immersion, and being smooth is like being a submersion (with a margin of interpretation error). From this perspective, being étale is the algebro-geometric analog of being a local isomorphism. However, being a local isomorphism in the Zariski topology is significantly stronger; in many ways, this is part of why étale cohomology is used as a replacement of ordinary "topological" cohomology for varieties/schemes.

Definition 1.2. Let X be a scheme, and $\mathrm{Et}(X)$ the category of étale X-schemes. The *small étale site of* X is the category $\mathrm{Et}(X)$ (in which all maps are étale), and coverings $\{U_i \to U\}$ are jointly surjective families of étale morphisms over X [Sta22, 021B].

With this, we can consider the category $\mathbf{Sh}_{et}(X)$ of abelian sheaves on the small étale site of X; this is an abelian category, and the right derived functors of the (left exact) global sections functor $\Gamma(X, -)\colon \mathbf{Sh}_{et}(X) \to \mathbf{Ab}$ are the *étale cohomology groups.*

Example 1.1. If Z is an X-scheme, then it represents a presheaf $\mathscr{F}(U) = \mathrm{Hom}_X(U, Z)$; it turns out this is actually a sheaf (in other words, the étale topology is *subcanonical*). As an example, consider the varieties μ_n (defined by $x^n - 1 = 0$) or $\mathbf{G}_m$, the punctured affine line. These define étale sheaves $\mu_n, \mathbf{G}_m$, with $\mu_n(U)$ being the nth roots of unity of 1 in $\Gamma(U, \mathscr{O}_U)$ and $\mathbf{G}_m(U) = \Gamma(U, \mathscr{O}_U)^\times$ [Mil13, 6.10]. The *Kummer sequence* relates these via

$$0 \to \mu_n \to \mathbf{G}_m \xrightarrow{\times n} \mathbf{G}_m \to 0,$$

which can be used to compute the cohomology of curves. In fact, it turns out that in general $H^0(X_{et}, \mathbf{G}_m) = \Gamma(X, \mathscr{O}_X)^\times$ and $H^1(X_{et}, \mathbf{G}_m) = \mathrm{Pic}(X)$.

As an example of why étale cohomology is the "right" notion cohomology for schemes, consider the case of curves:

Proposition 1.3 (Étale cohomology of curves)**.** *Let X be a connected complete algebraic curve over a separably closed field k. Then for n relatively prime to the characteristic of k,*

$$H^q(X_{et}, \mu_n) = \begin{cases} \mu_n(k) & q = 0 \\ {}_n \operatorname{Pic}(X) & q = 1 \\ \operatorname{Pic}(X)_n & q = 2 \\ 0 & q > 2, \end{cases}$$

where ${}_n \operatorname{Pic}(X)$ and $\operatorname{Pic}(X)_n$ are the kernel (resp. cokernel) of the multiplication by n map on $\operatorname{Pic}(X)$*. If X is smooth, then ${}_n \operatorname{Pic}(X) \simeq (\mathbf{Z}/n\mathbf{Z})^{2g}$ and $\operatorname{Pic}(X)_n \simeq \mathbf{Z}/n\mathbf{Z}$, where g is the genus of X* [Tam94, 10.3.5].

Under these assumptions, we also have an isomorphism $\mu_n \simeq \underline{\mathbf{Z}/n\mathbf{Z}}$, where the RHS is the constant sheaf, given by choosing a primitive root of unity. Thus, this computation shows that the ranks of the cohomology groups $H^q(X_{et}, \underline{\mathbf{Z}/n\mathbf{Z}})$ are 1, $2g$, and 1, just as in the case of the topological cohomology Riemann surfaces.

Workhorse theorems of étale cohomology. To conclude this flyby of étale cohomology, we record some useful theorems. First, consider a pullback diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

For any abelian sheaf $\mathscr{F}$ on X_{et}, $q \geq 0$, there is a base change morphism

$$g^*(R^q f_* \mathscr{F}) \to R^q f'_*({g'}^* \mathscr{F}).$$

Theorem 1.4 (Base Change for Smooth/Proper Morphisms)**.** *Let either of the two conditions be satisfied:*

(1) f is a proper morphism and $\mathscr{F}$ is a torsion sheaf.

(2) f is qcqs, g is smooth, and $\mathscr{F}$ is ℓ-torsion with ℓ invertible on Y.

Then the base change morphism is an isomorphism [Tam94, 11.3.1,5].

Corollary 1.5. *Let $k \subset k'$ be separably closed fields, X proper over k, and $X' = X \times_k k'$. Then for $\mathscr{F}$ a torsion sheaf on X_{et}, $q \geq 0$,*

$$H^q(X, \mathscr{F}) \simeq H^q(X', \mathscr{F}'),$$

with $\mathscr{F}'$ the pullback of $\mathscr{F}$ under $X' \to X$ [Tam94, 11.3.4].

Finally, we have the following finiteness theorem.

Theorem 1.6 (Finiteness)**.** *Let X be a smooth algebraic scheme over a separably closed field k, $\mathscr{F}$ a locally constant finite abelian sheaf of order coprime to the characteristic of k. Then the groups $H^q(X_{et}, \mathscr{F})$ are finite for all $q \geq 0$* [Tam94, 11.4.3].

2. The Weil Conjectures

As an illustration of one of the greatest successes of étale cohomology, we briefly discuss the Weil conjectures, a family of statements on the number-theoretic content of algebraic varieties, whose original proof by Deligne relied on étale cohomology. Much of this presentation comes from [Mil13] and [FK88].

Let X be a nonsingular projective variety over $\mathbf{F}_q$, for $q = p^a$ with p prime. For each $m = 1, 2, \ldots$, let N_m be the number of $\mathbf{F}_{q^m}$-rational points of X.

Definition 2.1. The *zeta function* of X is the formal power series

$$Z_X(t) = \exp\left(\sum_{m\geq 1} N_m \frac{t^m}{m}\right),$$

i.e. with

$$\frac{d}{dt}\log Z_X(t) = \sum_{m\geq 1} N_m t^{m-1}$$

being the generating function for $N_1, N_2, \ldots$.

The Weil conjectures are about controlling the behavior of this zeta function, and thus getting a grasp on the arithmetic data it contains.

Theorem 2.2 (Weil Conjectures). *Let X have dimension d, and let χ be the Euler-Poincaré characteristic of $X_{\overline{\mathbf{F}_q}}$, for a fixed choice of algebraic closure $\overline{\mathbf{F}_q}$.*

(a) (Rationality) $Z_X(t)$ is a rational function of t.

(b) (Functional equation) We have $Z(1/q^d t) = \pm q^{d\chi/2} t^{\chi} Z(t)$.

(c) (Riemann hypothesis) We can write

$$Z_X(t) = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\cdots P_{2d}(t)},$$

with $P_0(t) = 1 - t$, $P_{2d}(t) = 1 - q^d t$, and for $r = 1, \ldots, 2d-1$,

$$P_r(t) = \prod_{i=1}^{\beta_r}(1 - \alpha_{r,i} t),$$

with $\alpha_{r,i}$ algebraic integers of absolute value $q^{r/2}$.

(d) (Betti numbers) The degrees β_r of $P_r(t)$ are equal to the classical Betti numbers of $X_{\overline{\mathbf{F}_q}}$, considered as a complex variety.

Remark 2.1. To see why (c) is called the Riemann hypothesis, let us define the *zeta function* of a finite type $\mathbf{Z}$-scheme Y via

$$\zeta_Y(s) = \prod_{y \text{ closed}} \frac{1}{1 - N(y)^{-s}},$$

where $N(y)$ is the order of $\kappa(y)$, which is finite iff y is closed. For $\Re(s) > \dim Y$ this product coverges, giving a holomorphic function. For $Y = \operatorname{Spec} Z$, this is Riemann's zeta function. If we regard X as above as a finite type $\mathbf{Z}$-scheme, then some algebraic manipulation shows that

$$Z_X(t) = \prod_{x \text{ closed}} \frac{1}{1 - t^{\deg x}},$$

so $\zeta_X(s) = Z_X(q^{-s})$. The Riemann hypothesis for X says that the zeros of $\zeta_X(s)$ are s satisfying $1 = \alpha q^{-s}$, with $|\alpha q^{-s}| = q^{r/2-s}$ with $r = 1, \dots, 2d - 1$, and likewise for poles. Thus, the zeros of $\zeta_X(s)$ lie on the lines $\Re(s) = \frac{1}{2}, \frac{3}{2}, \dots, \frac{2d-1}{2}$, and its poles on the lines $\Re(s) = 0, 1, \dots, d$. [Mil13, 26.1].

On a historical note, Weil did not make conjectures lightly; they were originally conjectured in a paper full of computations verifying these conjectures in simple cases, and he proved them for the case of curves, which we give a short account of later [Wei49]. The form of these conjectures is derived largely from the ideas of algebraic topology, and the use of cohomology to dualize cocycles or count fixed points: a key observation here is that if $\overline{X} := X_{\overline{\mathbf{F}_q}}$ and F is the $\overline{\mathbf{F}_q}$-linear qth power Frobenius on $\overline{X}$, then the $\mathbf{F}_{q^n}$-rational points of X correspond to the fixed points of F^n. This is where the notion of a "Weil cohomology theory" comes from – a cohomology theory for algebraic varieties which is refined enough and "algebraic-topological" enough to prove these conjectures.

The Weil conjectures can be proved through a version of étale cohomology known as ℓ-adic cohomology: despite most of the theorems of étale cohomology being stated with finite coefficients, this is an extension to characteristic 0.

Definition 2.3. An ℓ-adic sheaf on X is an inverse system $(\mathscr{F}_n, f_{n+1}\colon \mathscr{F}_{n+1} \to \mathscr{F}_n)$ with

(a) each $\mathscr{F}_n$ a constructible sheaf of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules, and

(b) f_{n+1} inducing an isomorphism $\mathscr{F}_{n+1}/\ell^n\mathscr{F}_{n+1} \to \mathscr{F}_n$.

For such an ℓ-adic sheaf $\mathscr{F} = (\mathscr{F}_n)$, the *$\ell$-adic cohomology* of X is

$$H^*(X_{et}, F) = \varprojlim H^*(X_{et}, \mathscr{F}_n).$$

Beware that this is *not* the same as $H^*(X_{et}, \varprojlim \mathscr{F}_n)$ [Mil13, 19].

For example, we have

$$H^*(X_{et}, \mathbf{Z}_\ell) = \varprojlim H^*(X_{et}, \mathbf{Z}/\ell^n\mathbf{Z}),$$

and can extend this to field coefficients via $H^*(X, \mathbf{Q}_\ell) = H^*(X, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. In particular, this ℓ-adic cohomology supports a version of Poincaré duality and a Lefschetz theorem:

Theorem 2.4 (Poincaré Duality)**.** *Let $\mathscr{F} \otimes \mathbf{Q}_\ell$ be a locally constant locally free constructible ℓ-adic sheaf. Then, there exists a natural nondegenerate pairing*

$$H^p\left(X, \operatorname{Hom}(\mathscr{F} \otimes \mathbf{Q}_\ell, \mathbf{Q}(d))\right) \times H_c^{2d-p}(X, \mathscr{F} \otimes \mathbf{Q}_\ell) \to \mathbf{Q}_\ell.$$

Here the second factor is cohomology with compact support (i.e. $R\Gamma_c(X_{et}, -)$) [FK88, II.1.17].

Theorem 2.5 (Lefschetz Fixed-Point Theorem)**.** *Let X be a complete nonsingular variety over an algebraically closed field $k = \overline{k}$, and $\varphi\colon X \to X$ a regular map. Then, if Γ_φ is the graph of φ and Δ is the diagonal in $X \times X$, we have*

$$\Gamma_\varphi \cdot \Delta = \sum (-1)^r \operatorname{Tr}(\varphi \mid H^r(X, \mathbf{Q}_\ell)).$$

See [Mil13, 25.1] *or* [FK88, II.2.9].

These two theorems go a long way to proving the Weil conjectures: the functional equation is a consequence of Poincaré duality, and rationality and Betti numbers can be deduced from the Lefschetz theorem. More precisely, one can show that if X has dimension d, then

$$Z_X(t) = \frac{P_1(t)\cdots P_{2d-1}(t)}{P_0(t)\cdots P_{2d}(t)}$$

with $P_r(t) = \det\left(1 - Ft \mid H^r(X_0, \mathbf{Q}_\ell)\right)$ [Mil13, 27.6], with the functional equation and Betti number conjecture also holding.

Originally, Grothendieck made the step of using étale cohomology and these algebraic topology-inspired tools to prove all the Weil conjectures but the Riemann hypothesis. That result is due to Deligne, who used techniques from representation theory and the geometry of "Lefschetz pencils" to derive estimates on the $|\alpha_{r,i}|$'s. However, Weil knew this result for curves much earlier, and it is this simpler proof of the Riemann hypothesis we present below.

3. Weil's Proof for Curves

This section outlines Weil's original proof of the Riemann hypothesis for curves, using the much simpler and classical machinery of intersection theory on surfaces. My presentation draws from [Ji21] and exercises V.1.9 and V.1.10 in [Har77]. The place we start is the following: we have

$$Z_X(t) = \frac{\prod_{i=1}^{2g}(1 - \alpha_i t)}{(1-t)(1-qt)},$$

satisfying the functional equation $Z_X(1/qt) = q^{1-g}t^{2-2g}Z_X(t)$, with the α_i being algebraic integers in conjugate pairs with $\prod_{i=1}^{2g} a_i = q^g$. To prove the Riemann hypothesis, which in this case means $|\alpha_i| = \sqrt{q}$, we first prove an intermediate *Hasse-Weil bound.*

Some recollections from intersection theory. We'll need the following two results from intersection theory on surfaces; recall that we have a bilinear intersection form $(- \cdot -)$ from the group of divisors on a surface X to $\mathbf{Z}$, which, for curves C, D on X sharing no irreducible components, recovers the classical notion of $C \cdot D = \#(C \cap D)$. In fact, in this case, we can show that $C \cdot D = \deg_C \mathcal{O}(D)|_C$ [Vak22, 21.2.A].

Lemma 3.1. *Let C be a curve on a surface X. Then the self-intersection number $C^2 = C \cdot C$ is $\deg_C \mathcal{N}_{C/X}$.*

Proof. By definition, for $\mathcal{I} = \mathcal{O}_X(-C)$ being the ideal sheaf of C in X, the conormal sheaf is $\mathcal{I}/\mathcal{I}^2|_C$, which by [Vak22, 22.2.H] is isomorphic to $\mathcal{O}(-C)|_C$. The degree of the dual of this sheaf, $\mathcal{O}(C)|_C$, is precisely C^2. □

Next, we have the Hodge index theorem:

Theorem 3.2 (Hodge Index Theorem)**.** *Let H be an ample divisor on a surface X, and suppose D is a numerically nontrivial divisor with $D \cdot H = 0$. Then $D^2 < 0$* [Har77, V.1.9].

This can be weakened to the statement that if D is any divisor with $D \cdot H = 0$, then $D^2 \leq 0$. Combining these allow us to prove a useful estimate:

Lemma 3.3. *Let $X = C \times C'$ be the product of two curves, and $l = [C \times *]$, $m = [* \times C']$. If D is a divisor on X, $a = D \cdot l$, $b = D \cdot m$, then $D^2 \leq 2ab$.*

Proof. Consider the divisor $H = l + m$, which is ample by Nakai-Moishezon [Har77, V.1.10], and $E = l - m$. We have $H^2 = 2$, $E^2 = -2$, and $H \cdot E = 0$.

$$D' := (H^2)(E^2)D - (E^2)(D \cdot E)H - (H^2)(D \cdot H)E.$$

We compute that $H \cdot D' = (H^2)(D \cdot H)(E \cdot H) = 0$, so by 3.2

$$\begin{aligned} D' \cdot D' &= (H^2)^2(E^2)^2(D^2) + (E^2)^2(D \cdot E)^2(H^2) + (H^2)^2(D \cdot H)^2(E^2) \\ &\quad - (H^2)(E^2)^2(D \cdot E)(D \cdot H) - (H^2)^2(E^2)(D \cdot H)(D \cdot E) \\ &= 16D^2 + 8(a-b)^2 - 8(a+b)^2 - 8(a-b)(a+b) + 8(a-b)(a+b) \\ &= 16\left(D^2 + \frac{1}{2}\left((a-b)^2 - (a+b)^2\right)\right) \\ &= 16\left(D^2 - 2ab\right) \leq 0 \\ \implies D^2 &\leq 2ab, \end{aligned}$$

thus completing the proof [Har77, Exercise V.1.9]. □

The Hasse-Weil Bound. Now we apply the previous estimate to divisors on the surface $Y = X \times X$ (here we consider X and all products over $\overline{\mathbf{F}_q}$). Let Δ be the diagonal $X \hookrightarrow Y$ and Γ be the graph of the qth power relative Frobenius F over $\overline{\mathbf{F}_q}$. Note that if $f = F \times 1 \colon Y \to Y$, then $\Gamma = f^*\Delta$. Finally, let N be the number of $\mathbf{F}_q$-rational points of X: the Hasse-Weil bound aims to control the error term between N and $q + 1$.

Lemma 3.4. *With the notation as above, and letting g denote the genus of X, we have*

(a) $\Delta^2 = 2 - 2g$,

(b) $\Delta \cdot \Gamma = N$,

(c) $\Gamma^2 = q(2 - 2g)$.

Proof. (a) Note that if $\mathscr{I}$ is the ideal sheaf of Δ, then $\mathscr{I}/\mathscr{I}^2|_X \simeq \Omega^1_X$ (almost by definition). Thus, $\Delta^2 = \deg(\Omega^1_X)^\vee = -(2g-2) = 2-2g$.

(b) The fixed points of F on X are precisely the $\mathbf{F}_q$-rational points of X; to see that $\Delta \cdot \Gamma$ gives this number, we need to check that the intersection points are reduced. On define space, this means checking that

$$\prod_{i=1}^{k} \operatorname{Spec} \mathbf{F}_q[x_i]/(x_i^q - x_i)$$

is reduced. Taking derivatives, we see that $dx_i = 0$, so this holds.

(c) Finally, by [Vak22, 21.1.J], we have $\Gamma \cdot \Gamma = f^*\Delta \cdot f^*\Delta = (\deg f)(\Delta \cdot \Delta) = q(2-2g)$ □

Also, if $l = X \times *$ and $m = * \times X$ for $* \in X$, then $\Delta \cdot l = \Delta \cdot m = 1$ and $\Gamma \cdot l = q$, $\Gamma \cdot m = 1$.

Proposition 3.5 (Hasse-Weil Bound)**.** *In the notation above, we have*

$$|N - q - 1| \leq 2g\sqrt{q}.$$

Proof. Consider the divisor $D = r\Gamma + s\Delta$, which has

$$D^2 = r^2 q(2-2g) + 2rsN + s^2(2-2g).$$

Moreover, $D \cdot l = rq + s$ and $D \cdot m = r + s$. By 3.3, we get

$$r^2 q - r^2 qg + rsN + s^2 - s^2 g \leq (rq+s)(r+s) = r^2 q + rs(q+1) + s^2.$$

Rearranging gives $rs(N-q-1) \leq r^2 qg + s^2 g$. Assume that $r \neq 0, s \neq 0$; in either case the resulting bound is useless. Depending on if rs is positive or negative, dividing gives

$$|N - q - 1| \leq g \left| \frac{rq}{s} + \frac{s}{r} \right|.$$

Now, letting $x = r/s \in \mathbf{Q} - \{0\}$, we see that $|xq + x^{-1}|$, as a function of x, has minimum $2\sqrt{q}$. Ergo, the RHS can be made arbitrarily close to $2\sqrt{q}$, thus completing the proof. □

The Riemann hypothesis for curves. With this in hand, we can complete the proof of the Riemann Hypothesis for curves. Recall that we have expressed the zeta function as

$$Z_X(t) = \frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-qt)},$$

with the α_i algebraic integers in conjugate pairs, and $\prod_{i=1}^{2g} \alpha_i = q^g$. Thus, to show $|\alpha_i| = \sqrt{q}$ as desired, it suffices to show that $|\alpha_i| \leq \sqrt{q}$.

Lemma 3.6. *Let N_m be the number of $\mathbf{F}_q^m$-rational points of X. Then $a_m := N_m - q^m - 1$ satisfies $|a_m| \leq 2g\sqrt{q^m}$ and $a_m = \sum_{i=1}^{2g} \alpha_i^m$.*

Proof. The first claim follows immediately from 3.5. Next, we know that

$$\begin{aligned}
\sum_{m\geq 1} N_m t^{m-1} &= \frac{d}{dt}\log Z_X(t) \\
&= \frac{d}{dt}\left(\sum_{i=1}^{2g}\log(1-\alpha_i t) - \log(1-t) - \log(1-qt)\right) \\
&= -\sum_{i=1}^{2g}\frac{\alpha_i}{1-\alpha_i t} + \frac{1}{1-t} + \frac{q}{1-qt} \\
&= \sum_{m\geq 1}\left(1 + q^m + \sum_{i=1}^{2g}\alpha_i^m\right)t^{m-1}.
\end{aligned}$$

Comparing coefficients, we conclude that $a_m = \sum_{i=1}^{2g} \alpha_i^m$, as desired. □

Proposition 3.7 (RH for curves)**.** *For each i, $|\alpha_i| \leq \sqrt{q}$.*

Proof. For the sake of contradiction, assume WLOG that $|\alpha_1| = \max_i |\alpha_i| > \sqrt{q}$. Then, write

$$\sum_{m\geq 1} a_m t^m = \sum_{i=1}^{2g}\frac{\alpha_i t}{1-\alpha_i t}.$$

Note that the LHS converges absolutely on the disc $|t| < q^{-1/2}$, as here we have

$$\sum_{m\geq 1} |a_m t^m| < \sum_{m\geq 1} 2g|q^{m/2}t^m|,$$

where the latter sum converges for $|q^{1/2}t| < 1$. Thus, as $t \to \alpha_1^{-1}$, we have $|a_m t^m| \to |a_m \alpha_1^{-m}| < 2g|q^{m/2}q^{-m/2}| = 2g$, so the LHS converges. However, the RHS diverges as $\alpha_1 t \to 1$, which is a contradiction. Therefore, $|\alpha_i| \leq \sqrt{q}$, thus completing the proof. □

As a quasimeaningful final meditation, note that intersection theory is a cornerstone of algebraic topology; the cup product in cohomology is Poincaré dual to the intersection product, so in some sense, the ring structure on cohomology comes from intersection theory. In a similar way as the above proof uses estimates on the intersection number of various divisors on a surface, much of the theory of Poincaré duality for étale cohomology is about turning subvarieties into cohomology classes to get a better grasp on intersections for codimension > 1. To quote the person who got me into algebraic geometry, "motives, and basically everything, is secretly intersection theory."

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