THH AND BÖKSTEDT PERIODICITY

RUSHIL MALLARAPU

Contents

Hochschild homology and topological Hochschild homology are computable invariants of rings which are of great relevance to the study of algebraic K-theory. A landmark computational result in this area is Bökstedt periodicity, which states that $\mathrm{THH}(\mathbf{F}_p)$ is a polynomial algebra, in constrast to $\mathrm{HH}(\mathbf{F}_p)$ being a divided power algebra. We introduce both classical and topological Hochschild homology, and give an exposition of this result, using known computations of the Dyer-Lashof operations on the dual Steenrod Algebra.

This is an end-of-semester report for the Fall 2022 Directed Reading Program, with Natalie Stewart. No background on Hochschild homology is assumed, although some familiarity with stable homotopy theory is required for the section on THH. All mistakes are my own; please reach out to me if you spot anything!

1. Classical Hochschild Homology

To start, we define Hochschild homology over $\mathbf{Z}$ and compute $\mathrm{HH}(\mathbf{F}_p)$.

1.1. Hochschild Homology and the HKR Isomorphism.

Let R be an associative, unital ring. Consider the simplicial ring $B(R)_*$ with $B(R)_n := R^{\otimes_{\mathbf{Z}} n+1}$ and face maps

$$d_i(r_0 \otimes \cdots \otimes r_n) = \begin{cases} \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n & i < n \\ r_n r_0 \otimes \cdots \otimes r_{n-1} & i = n \end{cases}.$$

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Definition 1.1. The *Hochschild homology groups of* R, $\mathrm{HH}_*(R)$, are the homotopy groups of this SCR. By Dold-Kan, this is equivalently the homology of the chain complex:

$$\mathrm{HH}(R) = \cdots \to R \otimes_{\mathbf{Z}} R \otimes_{\mathbf{Z}} R \otimes_{\mathbf{Z}} R \to R \otimes_{\mathbf{Z}} R \otimes_{\mathbf{Z}} R \to R \otimes_{Z} R \to R,$$

with differential $d = \sum_{i=0}^{n} d_i$.

Note that if R is commutative, then the Hochschild complex $\mathrm{HH}(R)$ is associated to a simplicial commutative ring, giving it the structure of a strict CDGA.[1]

One classical source of interest in Hochschild homology comes from its connections to differential information. Recall the R-module of *absolute differential 1-forms* Ω^1_R. This is characterized by the following universal property. Recall that a derivation on R is a homomorphism from $R \to M$, for M an R-module such that the Leibniz rule holds: $d(rs) = rd(s) + sd(r)$. Ω^1_R is the target of the universal derivation $d\colon R \to \Omega^1_R$. This gives a presentation of Ω^1_R as the free R-module on symbols dr for each $r \in R$, modulo the relations

$$d(1) = 0, \quad d(r+s) = dr + ds, \quad d(rs) = rd(s) + sd(r),$$

with the universal derivation sending r to dr.

We can extend the module of differential 1-forms to a complex by defining higher differential forms as the exterior powers $\Omega^n_R := \bigwedge^n_R \Omega^1_R$, and using the induced differentials, organize this into the *de Rham complex*

$$\Omega^0_R \to \Omega^1_R \to \Omega^2_R \to \cdots.$$

Wedge product of forms gives the de Rham complex a CDGA structure [Sta22, 0FKL].

When R is commutative, the first Hochschild differential is 0, so we find that $\mathrm{HH}_0(R) = \Omega^0_R = R$. Furthermore, the map

$$\mathrm{HH}_1(R) = \frac{R^{\otimes 2}}{\mathrm{im}(R^{\otimes 3} \to R^{\otimes 2})} = \frac{R^{\otimes 2}}{\langle x_0x_1 \otimes x_2 - x_0 \otimes x_1x_2 + x_2x_0 \otimes x_1 \rangle} \to \Omega^1_R$$

given by $[x \otimes y] \mapsto x\, dy$ is an isomorphism via the obvious inverse; this quotient is precisely imposing the Liebniz rule. These isomorphisms hint at the following connection between Hoschchild homology and de Rham cohomology.

Theorem 1.2 (Hochschild-Kostant-Rosenberg)**.** *For a commutative ring R smooth over $\mathbf{Z}$,*

$$\Omega^*_R \to \mathrm{HH}_*(R)$$

is an isomorphism of graded algebras.

Remark 1.3. Note that there is no differential on $\mathrm{HH}_*(R)$ corresponding to the de Rham differential; such an operation can be constructed and is central to the story of negative cyclic and periodic homology, but is not relevant to what follows. In addition, there is a derived enhancement of this theorem giving an isomorphism between derived Hoschchild homology and the de Rham-Witt complex.

[1] Here, "strict" means that odd-degree elements square to zero – this is only important over $\mathbf{F}_2$ [NK, 2.3].

1.2. **Computing** $\mathrm{HH}(\mathbf{F}_p)$.

The statements of the previous section are fine so long as R is flat over $\mathbf{Z}$. However, when this is not the case, we need to derive everything in sight. Thus, from here on, we redefine the Hochschild complex as

$$\mathrm{HH}(R) := R \otimes^L_{R\otimes^L R^{\mathrm{op}}} R.$$

Using the bar resolution of R as an $R-R$ bimodule, it is easy to see that is equivalent to the previously presented complex for flat rings.

Now, we can compute $\mathrm{HH}(\mathbf{F}_p)$. To start, we can compute $\mathbf{F}_p \otimes^L \mathbf{F}_p$ via the flat resolution

$$\mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{F}_p,$$

corresponding to the CDGA $\mathbf{Z}[t]/t^2$, with $dt = p$ and $|t| = 1$. Tensoring with $\mathbf{F}_p$, we get

$$\mathbf{F}_p \otimes^L \mathbf{F}_p \simeq \mathbf{F}_p[t]/t^2 =: S.$$

From here, we need to resolve $\mathbf{F}_p$ as an S-algebra. One way to do so is by the divided power CDGA

$$S\langle x\rangle = \frac{S[x_1, x_2, \ldots]}{x_i x_j = \binom{i+j}{i} x_{i+j}} = \frac{\mathbf{F}_p[t, x_1, x_2, \ldots]}{t^2,\ x_i x_j = \binom{i+j}{i} x_{i+j}}, \quad dx_i = t x_{i-1}, \quad |x_i| = 2i.$$

Let us understand this CDGA: a direct computation shows that for $k > 0$

$$S\langle x\rangle_{2k} = \mathbf{F}_p\{x_k\} = \mathbf{F}_p\left\{x_1^k/k!\right\}, \text{ and } S\langle x\rangle_{2k+1} = \mathbf{F}_p\{t x_k\} = \mathbf{F}_p\left\{t x_1^k/k!\right\}.$$

The differential $S\langle x\rangle_{2k} \to S\langle x\rangle_{2k-1}$ is an isomorphism, as it sends the generator to the generator, and the differential coming from odd degrees is zero, as

$$d(t x_k) = t d(x_k) + x_k d(t) = t^2 x_{k-1} + p x_k = 0.$$

Therefore, we get a resolution of $\mathbf{F}_p$ over S. Tensoring with $\mathbf{F}_p$ over S has the effect of killing t, which collapses everything in even degrees. Thus,

$$\mathbf{F}_p \otimes^L_S \mathbf{F}_p \cong \frac{\mathbf{F}_p[x_1, x_2, \ldots]}{x_i x_j = \binom{i+j}{i} x_{i+j}}$$

with zero differentials. We have proved:

Proposition 1.4. *The Hochschild homology* $\mathrm{HH}_*(\mathbf{F}_p) = \mathbf{F}_p[x, x^2/2!, x^3/3!, \ldots] = \mathbf{F}_p\langle x\rangle$ *is the free divided power algebra on a generator in degree 2.*

2. Topological Hochschild Homology

2.1. **Base-Changing to the Sphere.** The previous result is a delightful computation, but there is something unsatisfying about divided power algebras. Stable homotopy theory tells us that we can seek enlightenment by switching to a deeper base than $\mathbf{Z}$, namely the sphere spectrum. Thus, identifying discrete (i.e. "classical") rings with $\mathbf{Z}$-algebras[2] in the symmetric monoidal∞-category of spectra $\mathbf{Sp}$, we can give the following definition:

Definition 2.1. For R in $\mathbf{Alg}(\mathbf{Sp})$, the *topological Hochschild homology* of R is

$$\mathrm{THH}(R) := R \otimes_{R \otimes_{\mathbb{S}} R^{\mathrm{op}}} R$$

where the tensor product is taken in $\mathbf{Sp}$. The *topological Hochschild homology group* $\mathrm{THH}_*(R)$ *of* R are the homotopy groups of this spectrum.

There is a relative version of THH for an S-algebra R given by $\mathrm{THH}(R/S) = R \otimes_{R \otimes_S R^{\mathrm{op}}} R$; thus (derived) Hochschild homology is simply $\mathrm{HH}(R) = \mathrm{THH}(R/\mathbf{Z})$.

In this context, we have a fundamental result of Bökstedt:

Theorem 2.2. *There is an isomorphism of algebras* $\mathrm{THH}_*(\mathbf{F}_p) \cong \mathbf{F}_p[x]$, *with* $|x| = 2$.

Thus, as expected, topological Hochschild homology is a much better invariant in characteristic p than classical Hochschild homology.

2.2. **Proof of Bökstedt Periodicity.** A good reference for this section is [KN19] or [Zha20]. The actual result we will prove is the following.

Theorem 2.3. *As an* $\mathbf{E}_1$*-*$\mathbf{F}_p$*-algebra spectrum,* $\mathrm{THH}(\mathbf{F}_p)$ *is equivalent to* $\mathbf{F}_p \otimes_{\mathbb{S}} \Sigma^\infty_+ \Omega S^3$.

This directly implies Theorem 2.2 as the homology of ΩS^3 is the Pontryagin ring, which is polynomial on a degree 2 generator. This result relies on the following structural result, equivalent to the Hopkins-Mahowald theorem on the free $\mathbf{E}_2$-$\mathbf{F}_p$-algebra being $\mathbf{F}_p$.[3]

Lemma 2.4. *As an* $\mathbf{E}_2$*-*$\mathbf{F}_p$*-algebra, the dual Steenrod algebra* $\mathbf{F}_p \otimes_{\mathbb{S}} \mathbf{F}_p$ *is free on a single generator in degree 1; it is* $\mathbf{E}_2$*-*$\mathbf{F}_p$*-equivalent to* $\mathbf{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3$

To show this, we need the following proposition:

Proposition 2.5. *Let* R *be the free* $\mathbf{E}_2$*-*$\mathbf{F}_p$*-algebra on a generator in degree 1. Then*

(a) For $p = 2$, $\pi_* R = \mathbf{F}_2[x_1, x_2, \ldots]$, *with* $|x_i| = 2^i - 1$. *The Dyer-Lashof operations are given by* $x_{i+1} = Q^{2^i} x_i$ *and* $\beta x_i = x_{i-1}^2$.

(b) For $p > 2$, $\pi_* R = \bigwedge_{\mathbf{F}_p}(y_0, y_1, \ldots) \otimes \mathbf{F}_p[z_1, z_2, \ldots]$, *with* $|y_i| = 2p^i - 1$, $|z_i| = 2p^i - 2$, *and Dyer-Lashof operations* $y_{i+1} = Q^{p^i} y_i$ *and* $z_i = \beta y_{i+1}$.

[2] This identification is given by sending a ring R to it's Eilenberg-Mac Lane spectrum, also denoted R; it is known that discrete rings are canonically lifted to spectra as $\mathbf{E}_1$-$\mathbf{Z}$-algebras. In fact, there is an equivalence between $\mathbf{Mod}_{\mathbf{Z}}$, the category of $\mathbf{Z}$-modules in $\mathbf{Sp}$, and $D(\mathbf{Z})$.

[3] For a great discussion of the equivalence, see [KN19, A].

Any $\mathbf{E}_2$*-*$\mathbf{F}_p$*-algebra* R *with* $\pi_* R$ *satisfying the above is also free on a generator in degree 1.*

Proof. Note that the free $\mathbf{E}_2$-algebra on S^1 is $\Omega^2 S^3$. Thus, $A = \mathbf{F}_p \otimes \Omega^2 S^3$, so the computation is that of the Pontryagin ring of $\Omega^2 S^3$ [KN19, 1.4].

Then, given an R as in the last statement and any nontrivial $x_1 \in \pi_1 R$, we get an $\mathbf{E}_2$-map $\mathrm{Free}_{\mathbf{E}_2}(x_1) \to R$. The ring structure and Dyer-Lashof operations generate everything from x_1, so as the $\mathbf{E}_2$-map preserves these operations, it is an equivalence. □

Thus, Lemma 2.4 reduces to showing that the dual Steenrod algebra $\mathbf{F}_p \otimes \mathbf{F}_p$ has the right ring structure and Dyer-Lashof operations. This is a classical calculation due to Milnor and Steinberger, respectively [BMMS86, 3.2.2-3.2.3]. With this in hand, the proof of Theorem 2.3 is direct:

Proof. By Lemma 2.4, we know that $\mathbf{F}_p$ is resolvable as an $\mathbf{F}_p \otimes_{\mathbb{S}} \mathbf{F}_p$-module via

$$\mathbf{F}_p \simeq \mathbf{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3 \otimes_{\Sigma^\infty_+ \Omega^2 S^3} \mathbb{S} \simeq (\mathbf{F}_p \otimes \mathbf{F}_p) \otimes_{\Sigma^\infty_+ \Omega^2 S^3} \mathbb{S}..$$

Thus, we have the following equivalences of $\mathbf{E}_1$-algebras:

$$\begin{aligned} \mathrm{THH}(\mathbf{F}_p) &= \mathbf{F}_p \otimes_{\mathbf{F}_p \otimes_{\mathbb{S}} \mathbf{F}_p} \mathbf{F}_p \simeq \mathbf{F}_p \otimes_{\mathbf{F}_p \otimes_{\mathbb{S}} \mathbf{F}_p} (\mathbf{F}_p \otimes \mathbf{F}_p) \otimes_{\Sigma^\infty_+ \Omega^2 S^3} \mathbb{S} \\ &\simeq \mathbf{F}_p \otimes_{\Sigma^\infty_+ \Omega^2 S^3} \mathbb{S} \simeq \mathbf{F}_p \otimes \mathbb{S} \otimes_{\Sigma^\infty_+ \Omega^2 S^3} \mathbb{S} \\ &\simeq \mathbf{F}_p \otimes \Sigma^\infty_+ (* \otimes_{\Omega^2 S^3} *) \simeq \mathbf{F}_p \otimes \Sigma^\infty_+ \Omega S^3, \end{aligned}$$

giving the desired result. □

As a bonus, this result, along with the machinery of the ring of spherical Witt vectors associated to a perfect $\mathbf{F}_p$-algebra k, allows one to deduce the following extension.

Corollary 2.6. *For a perfect* $\mathbf{F}_p$*-algebra* k*, we have* $\mathrm{THH}(k) \cong k[x]$*, with* $|x| = 2$*.*

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Email address: rushil_mallarapu@college.harvard.edu