ITÔ DIFFUSIONS AND THE DIRICHLET PROBLEM

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Contents

Stochastic processes are collections of random variables indexed by time, and their study is critical to statistics, physics, and finance. In particular, Itô diffusions, which arise as the stochastic integrals of time-homogenous functions, have a rich theory, coming from their connection to elliptic PDE. The goal of this talk is to explore this connection, and use these diffusions to explicitly solve some PDEs. We will first give an overview of the theory of Itô diffusions, discuss their associated mean-value properties and connections to elliptic operators, and finally leverage some hard theorems to solve the Dirichlet problem.

These are notes from my final presentation for Math 289Y, a Fall 2023 topics course on elliptic PDE taught by Freid Tong. All mistakes are my own; please reach out to me if you spot anything!

1. Brownian Motion and Itô Diffusions

1.1. Brownian motion and Itô integrals.

Definition 1.1. An *n-dimensional Brownian motion* is an $\mathbf{R}^n$-valued continuous-time stochastic process $\left\{B_t = (B_t^{(1)}, B_t^{(2)}, \ldots, B_t^{(n)})\right\}_{t\geq 0}$ satisfying

(a) (Continuity) The function $t \mapsto B_t$ is continuous.

(b) (Stationary increments) For $0 \leq s, t$, $B_{t+s} - B_s$ and B_t are identically distributed.

(c) (Independent increments) For $0 \leq r < s \leq t < u$, $B_u - B_t$ and $B_s - B_r$ are independent.

(d) (Normal) For $0 \leq t$, the random vector B_t is Multivariate Normal with mean vector 0 and covariance matrix tI.

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That Brownian motions, starting at given points $x \in \mathbf{R}^n$, even exist is a consequence of Kolmogorov's extension theorem, and we won't concern ourself with this technicality [Oks07, 2.1.5].

Let X_t be an $\mathbf{R}^n$-valued stochastic process, i.e. a continuous sequence of random vectors. The study of stochastic differential equations concerns such processes satisfying equations like

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot \text{“noise"},$$

where we have some random noise. Effectively, the only reasonable source of this random noise comes as the "formal derivative" of Brownian motion. Such a thing doesn't literally exist – Brownian motion is nowhere differentiable – but this ensures that the integrated noise has stationary, independent increments with mean 0. Thus, we can suggestively rewrite this stochastic differential equation as

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t.$$

Engaging in some wishful thinking, a solution of this SDE would satisfy

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s.$$

The definition of Itô integrals, i.e. expressions of the form $\int_0^t \sigma(s, X_s)\,dB_s$, is rigged to make sense of this intuition. What follows is a whirlwind overview of the key ideas: see [Oks07, 3,4] for proofs.

Definition 1.2. Let $\mathcal{V}(S, T)$ denote the class of functions $f(t, \omega)\colon [0, \infty) \times \Omega \to \mathbf{R}$ such that

(a) f is $\mathcal{B} \times \mathscr{F}$-measurable, with $\mathcal{B}$ the Borel σ-algebra on the half line;

(b) there exists an increasing family $\mathcal{H}_t, t \geq 0$ of σ-algebras such that B_t is a martingale with respect to $\mathcal{H}_t$ and f_t is $\mathcal{H}_t$-adapted;

(c) $E\left[\int_S^T f(t, \omega)^2\,dt\right] < \infty$.

These are *Itô integrable functions.* For each n, let $S = t_1, \ldots, t_{n+1} = T$ denote a partition of $[S, T]$ Then, the Itô integral of f is the limit

$$\int_S^T f(t, \omega)\,dB_t(\omega) = \lim_{n \to \infty} \sum_{j=1}^n f(t_j, \omega)(B_{t_{j+1}} - B_{t_j})(\omega),$$

as the partition width goes to 0.

Remark 1.1. In reality, this is more an intuition then a definition; the proper definition involves imitating the definition of the Lebesgue integral; first defining the Itô integral for functions which are elementary and then approximating functions in $\mathcal{V}$ by elementary ones.

Thus, the Itô integral is a random variable, with the following properties: let $f, g \in \mathcal{V}(0,T)$, and $0 \le S < U < T$.

(a) $\int_S^T f\, dB_t = \int_S^U f\, dB_t + \int_U^T f\, dB_t$.

(b) $\int_S^T (cf + g)\, dB_t = c\int_S^T f\, dB_t + \int_S^t g\, dB_t$.

(c) $E\left[\int_S^T f\, dB_t\right] = 0$ and $E\left[\left(\int_S^T f\, dB_t\right)^2\right] = E\left[\int_S^T f^2\, dt\right]$ (Itô isometry).

(d) Let $\mathscr{F}_t$ denote the σ-algebra generated by $B_s, s \le t$. Then, $t \mapsto \int_0^t f\, dB_t$ can be chosen to be continuous in t, and is a martingale with respect to $\mathscr{F}_t$.

Itô integrals can be extended to multiple dimensions similarly; if $v = (v^{ij}(t,\omega))$ is an $m \times n$ matrix of functions, with each entry being Itô integrable, and $B_t = (B_t^{(1)}, \dots, B_t^{(n)})$ denoting n-dimensional Brownian motion, then

$$\int_S^T v\, dB_t = \int_S^T \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{m1} & \dots & v_{mn} \end{pmatrix} \begin{pmatrix} dB_t^{(1)} \\ \vdots \\ dB_t^{(n)} \end{pmatrix}.$$

1.2. **Solutions of SDEs and diffusions.**

Definition 1.3. An *Itô process* or *stochastic integral* is a stochastic process of the form

$$X_t = X_0 + \int_0^t u(s,\omega)\, ds + \int_0^t v(s,\omega)\, dB_s,$$

with v Itô integrable. This can be written in *differential form*

$$dX_t = u\, dt + v\, dB_t.$$

In n-dimensions, an $\mathbf{R}^n$-valued Itô integral is given in differential form above, with $u = (u^i(s,\omega))$ and $v = (v^{ij}(s,\omega))$, $1 \le i \le n$, and $1 \le j \le m$, and B_t an m-dimensional Brownian motion.

Unlike Itô integrals themselves, Itô processes are closed under smooth maps; this is the content of the famous Itô lemma:

Lemma 1.4. *Let $dX(t) = u\, dt + v\, dB(t)$ be an n-dimensional Itô process,*

$$g(t,x) = (g_1(t,x), \dots, g_p(t,x))$$

a C^2 map from $[0,\infty) \times \mathbf{R}^n \to \mathbf{R}^p$. Then, $Y(t,\omega) = g(t, X(t))$ is a p-dimensional Itô process, with kth component

$$dY_k = \frac{\partial g_k}{\partial t}(t,X)\, dt + \frac{\partial g_k}{\partial x_i}(t,X)\, dX_i + \frac{1}{2}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)\, dX_i dX_j,$$

with $dB_i dB_j = \delta_{ij} dt$ and $dB_i dt = 0$.

Proof. See [Oks07, 4.2.1]. □

The final background ingredient we need is the following theorem, which guarantees the existence of strong solutions to SDEs with "bounded" coefficients.

Theorem 1.5 (Existence and uniqueness for stochastic differential equations)**.** *Let $T > 0$, $b\colon [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$, $\sigma\colon [0,T] \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$ be measurable satisfying:*

(a) There exists a constant C s.t.

$$|b(t,x)| + |\sigma(t,x)| \le C(1 + |x|),$$

for all $x \in \mathbf{R}^n$, $0 \le t \le T$.

(b) There exists a constant D s.t.

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le D|x - y|,$$

for all $x, y \in \mathbf{R}^n$, $0 \le t \le T$.

Then, for any $x \in \mathbf{R}^n$, the SDE

$$dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dB_t, \quad 0 \le t \le T, X_0 = x$$

has a unique t-continuous solution X_t such that it is $\mathscr{F}_t$-adapted and

$$E\left[\int_0^T |X_t|^2\,dt\right] < \infty.$$

Proof. See [Oks07, 5.2.1]. Key idea is Picard iteration with Itô integrals! □

2. Harmonic Measure and the Infinitesimal Generator

For this section, let X_t be an Itô diffusion in $\mathbf{R}^n$, which means it is given by

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t,$$

where the coefficients are independent of time and satisfy the conditions in 1.5, so that the above SDE can be solved. For any $x \in \mathbf{R}^n$, let Q^x denote the distribution of $\{X_t\}_{t\ge 0}$ starting at $X_0 = x$, and E^x denote expectation with respect to this measure. Diffusions are special in stochastic PDE because they satisfy a *Markov property*: intuitively, conditional on their present behavior, their future is independent of the past. We will define stopping times and state the strong Markov property, and use this to show how diffusions give rise to mean value theorems on arbitrary domains.

2.1. **The strong Markov property.**

Definition 2.1. Let $\{\mathcal{N}_t\}$ denote an increasing family of σ-algebras. A random variable $\tau\colon \Omega \to [0,\infty]$ is a *stopping time* with respect to $\{\mathcal{N}_t\}$ if

$$\{\tau \le t\} \in \mathcal{N}_t, \quad \text{for all } t \ge 0.$$

Example 2.1. First, if $\tau = t_0$ is constant, then τ is trivially a stopping time, as

$$\{\tau \le t\} = \begin{cases} \Omega & t_0 \le t \\ \varnothing & t_0 > t \end{cases}.$$

Second, if $\mathscr{M}_t$ is the σ-algebra generated by $X_s, s \le t$, and $H \subset \mathbf{R}^n$ is a Borel set, then

$$\tau_H = \inf\{t > 0 \mid X_t \notin H\}$$

is a stopping time; $\mathscr{M}_t = \cap_{s>t}\mathscr{M}_s$ and any Borel set can be approximated by closed sets [Oks07, 7.2.2]. This is the *first exit time* from H. Note that the "last exit time" would not be a stopping time, as it depends on the behavior of X_t into the future.

Theorem 2.2 (Strong Markov Property for Itô Diffusions)**.** *Let $\mathscr{F}_t$ denote the σ-algebra generated by $B_s, s \le t$, τ a stopping time with respect to $\mathscr{F}_t$ with $\tau < \infty$ a.s., and $\mathscr{F}_\tau$ the σ-algebra generated by $B_{s\wedge\tau}, s \ge 0$. Then if $f\colon \mathbf{R}^n \to \mathbf{R}$ is bounded and measurable,*

$$E^x[f(X_{\tau+h} \mid \mathscr{F}_\tau] = E^{X_\tau}[f(X_h)], \quad \textit{for all } h \ge 0.$$

Proof. See [Oks07, 7.2.4]. The main idea is that Brownian motion itself has a strong Markov property, and this gets bootstrapped up to arbitrary diffusions. □

For $t \ge 0$, the *shift operator* on M_∞-measurable functions is defined bys

$$\theta_t\left(g_1(X_{t_1})\cdots g_k(X_{t_k})\right) = g_1(X_{t_1+t})\cdots g_k(X_{t_k+t}),$$

with g_i Borel measurable, and extending by taking limits. Then, the strong Markov property can be expressed as

$$E^x[\theta_\tau \eta | \mathscr{F}_\tau] = E^{X_\tau}[\eta],$$

for all bounded $\eta \in \mathcal{H}$.

2.2. **Harmonic measure.** Let $G \subset\subset H$ be measurable with $\tau_H < \infty$ a.s. Q^x. If g is any bounded continuous function on $\mathbf{R}^n$, we know that the intermediate hitting time $\inf\{t > \tau_G \mid X_t \notin H\}$ is just the first hitting time τ_H, so $\theta_{\tau_G} g(X_{\tau_H}) = g(X_{\tau_H})$. Therefore,

$$E^x\left[g(X_{\tau_H})\right] = E^x\left[E^x\left[\theta_{\tau_G} g(X_{\tau_H}) \mid \mathscr{F}_\tau\right]\right] = E^x\left[E^{X_{\tau_G}}\left[g(X_{\tau_H})\right]\right] = \int_{\partial G} E^y\left[g\left(X_{\tau_H}\right)\right] \cdot Q^x(X_{\tau_G} \in dy).$$

Based on this, we can define the *harmonic measure of X on ∂G* as

$$\mu_G^x(F) = Q^x(X_{\tau_G} \in F),$$

for $x \in G$ and $F \subset \partial G$. Because any $f \in L^1(\mu_G^x)$ can be approximated by bounded continuous functions as above, this identity applies to all bounded measurable functions.

Definition 2.3. Let f be locally bounded, measurable on H. f is called *X-harmonic* if, for all $x \in H$ and bounded opens $U \subset\subset H$,

$$f(x) = E^x[f(X_{\tau_U})].$$

Example 2.2. Suppose φ is bounded measurable on ∂H, and let $u(x) = E^x[\varphi(X_{\tau_H})]$ for $x \in D$. Then, by the above calculation, for any $U \subset\subset H$, u has a mean-value property

$$u(x) = \int_{\partial U} u(y) d\mu_U^x(y) = E^x[u(X_{\tau_U})],$$

so it is X-harmonic!

2.3. **The characteristic operator of a diffusion.** So far, the analogy should start to unfold: just as classical/weak harmonic functions are those satisfying a mean-value property on balls, X-harmonic functions satisfy a mean-value property with respect to X. It turns out that requiring a $C^2(\Omega)$ function u to satisfy a mean-value property on all compactly supported balls in Ω implies $\Delta u = 0$ on Ω. Similarly, X-harmonic functions should satisfy a certain PDE, as we'll see.

Definition 2.4. The *characteristic operator* of X_t is defined by

$$\mathcal{A}f(x) = \lim_{U \downarrow x} \frac{E^x[f(X_{\tau_U})] - f(x)}{E^x[\tau_U]},$$

where the limit is taken over open sets $U_1 \supset U_2 \supset \dots$ with $\cap U_k = \{x\}$.

For C^2 functions, this operator acts via a linear second order partial differential operator:

Theorem 2.5. *If $f \in C^2$, then the limit $\mathcal{A}f$ exists for all $x \in \mathbf{R}^n$, and*

$$\mathcal{A}f = \sum_i b^i D_i f + \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)^{ij} D_{ij}^2 f.$$

Proof. See [Oks07, 7.5.4]. □

In fact, this operator gives a sort of "fundamental theorem of calculus" for diffusions, known as the Dynkin formula.

Lemma 2.6. *Let $f \in C_0^2(\mathbf{R}^n)$, τ a stopping time with $E^x[\tau] < \infty$. Then*

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau \mathcal{A}f(X_s)\, ds\right].$$

Also, note that if f is X-harmonic, then $\mathcal{A}f = 0$, by definition; thus, the real challenge is finding solutions to $\mathcal{A}f = 0$ with a desired regularity.

3. Existence and Regularity for the Dirichlet Problem

In this section, we'll consider an elliptic operator

$$L = b^i(x)D_i + a^{ij}(x)D^2_{ij}$$

on a domain Ω, and some boundary data $\varphi \in C^0(\partial\Omega)$. The corresponding Dirichlet problem is to find $u \in C^2(\Omega)$ such that

$$\begin{cases} Lu = 0 & \text{on } \Omega \\ \lim_{x \to y} u(x) = \varphi(y) & \text{for all } y \in \partial\Omega. \end{cases}.$$

We assume that we can choose $\sigma(x) \in \mathbf{R}^n \times \mathbf{R}^n$ such that $\frac{1}{2}\sigma(x)\sigma(x)^T = (a^{ij}(x))$ (e.g. if $a^{ij} \in C^2(\Omega)$ and has bounded second partial derivatives), and that b and σ satisfy the conditions of 1.5. Thus, we can find a diffusion X_t with

$$dX_t = b\,dt + \sigma\,dB_t,$$

for B_t n-dimensional Brownian motion, with characteristic operator $\mathcal{A} = L$.

3.1. **Regular boundary points and the stochastic Dirichlet problem.** Initially, let us try to solve the following, weaker *stochastic Dirichlet problem*: given φ bounded and measurable on $\partial\Omega$, find an X-harmonic function u on Ω with $\lim_{t \uparrow \tau_D} u(X_t) = \varphi(X_{\tau_\Omega})$ a.s. Q^x, for all $x \in \Omega$.

Theorem 3.1 (Solution of the stochastic Dirichlet problem)**.** *Let φ be bounded measurable on $\partial\Omega$.*

(a) (Existence) The function $u(x) := E^x[\varphi(X_{\tau_\Omega})]$ solves the stochastic Dirichlet problem.

(b) (Uniqueness) If g is a bounded function on Ω which is X-harmonic and has $\lim_{t \uparrow \tau_\Omega} g(X_t) = \varphi(X_{\tau_\Omega})$ a.s. Q^x, then $g(x) = u(x)$.

Proof. (a) It follows from 2.6 that u is X-harmonic. Thus, fix $x \in \Omega$, let $\{\Omega_k\}$ be an increasing sequence of opens $\Omega_k \subset\subset \Omega$ with $\cup_k \Omega_k = \Omega$. Define $\tau_k = \tau_{\Omega_k}$ and $\tau = \tau_D$. Then, by the strong Markov property,

$$u(X_{\tau_k}) = E^{X_{\tau_k}}[\varphi(X_\tau)] = E^x\left[\theta_{\tau_k}(\varphi(X_\tau)) \mid \mathcal{F}_{\tau_k}\right] = E^x[\varphi(X_\tau) \mid \mathcal{F}_{\tau_k}],$$

so $M_k := E^x[\varphi(X_{\tau_k}) \mid \mathcal{F}_{\tau_k}]$ is a bounded discrete martingale. By the martingale convergence theorem, we have

$$u(X_{\tau_k}) = E^x[\varphi(X_\tau) \mid \mathcal{F}_{\tau_k}] \to \varphi(X_\tau)$$

pointwise a.s. and in $L^p(Q^x)$ for all $p < \infty$. Moreover, because $N_t = u\left(X_{\max(\tau_k, \min(t, \tau_{k+1}))}\right) - u(X_{\tau_k})$ is a martingale with respect to $\mathcal{G}_t := \mathcal{F}_{\tau_k \vee (t \wedge \tau_{k+1})}$, the martingale inequality

$$Q^x\left[\sup_{\tau_k \le r \le \tau_{k+1}} |u(X_r) - u(X_{\tau_k})| > \epsilon\right] \le \frac{1}{\epsilon^2} E^x\left[|u(X_{\tau_{k+1}}) - u(X_{\tau_k})|^2\right] \to 0,$$

as $k \to \infty$ for all $\epsilon > 0$. Thus, u satisfies the stochastic boundary condition as desired.

(b) If g is such a harmonic function, then $g(x) = E^x\left[g(X_{\tau_k})\right]$, and the boundary condition implies

$$g(x) = \lim_{k\longrightarrow\infty} E^x[g(X_{\tau_k})] = E^x[\varphi(X_\tau)]$$

by bounded convergence. □

Thus, we can always solve the stochastic Dirichlet problem somewhat explicitly. The problem is that in general, the function u may not even be continuous, much less C^2! More worryingly for now, the boundary condition need not always hold:

Example 3.1. Let $X(t) = (X_1(t), X_2(t))$ solve $dX_1 = dt, dX_2 = 0$, so $X(t) = X(0) + (t, 0)$, $X(0) \in \mathbf{R}^2$, $t \geq 0$. Define Ω to be

$$\Omega = ((0,1) \times (0,1)) \cup ((0,2) \cup (0,1/2)),$$

and let φ be a continuous function on $\partial\Omega$ with $\varphi = 1$ on $\{1\}\times[1/2,1]$ and 0 on $\{2\}\times[0,1/2]$ and $\{0\} \times [0,1]$. Then,

$$u(t,x) = E^{t,x}[\varphi(X_{\tau_\Omega}] = \begin{cases} 1 & x \in (1/2,1) \\ 0 & x \in (0,1/2) \end{cases},$$

and for $x \in (1/2, 1)$

$$\lim_{t\longrightarrow 0^+} u(t,x) = 1 \neq \varphi(0,x).$$

See [?]oksendal.

However, if the boundary is nice enough, we can sidestep this issue. First, a lemma.

Lemma 3.2. *Let $H \in \cap_{t>0}\mathscr{M}_t$. Then $Q^x(H) = 0$ or 1.*

Proof. For η a bounded, M_∞-measurable r.v., we have by Feller continuity that [?]oksendal

$$\int_H \eta dQ^x = \lim_{t\longrightarrow 0}\int_H \theta_t\eta dQ^x = \lim_{t\longrightarrow 0}\int_H E^{X_t}[\eta]dQ^x = Q^x(H)E^x[\eta],$$

via approximating η by continuous functions. Then, if $\eta = I_H$, this gives $Q^x(H) = Q^x(H)^2$, whence the 0-1 law. □

Corollary 3.3. *If $y \in \mathbf{R}^n$, then $Q^y[\tau_\Omega = 0]$ is 0 or 1; a.a. paths X_t starting from y either stay within Ω for a positive time or instantly leave.*

Definition 3.4. A point $y \in \partial\Omega$ is *regular for* Ω if $Q^y[\tau_\Omega = 0]$.

This feels odd, because you might assume by symmetry that paths have even probabilities of staying or leaving a domain, but it turns out that for infinitessimal behavior of diffusions, this can't happen!

3.2. **The Dirichlet problem for uniformly elliptic operators.** While we can't solve the Dirichlet problem in general, let us try to find a solution $u \in C^2$ which has the right boundary data at all regular boundary points: this is the *generalized Dirichlet problem.* As a preliminary, it is the case that all Itô diffusions where $\sigma(x)$ has a bounded inverse and satisfying an integrability condition satisfy *Hunt's condition*: every semipolar set for X_t is polar [Oks07, 9.2.12]. A semipolar (resp. polar) set is a measurable $G \subset \mathbf{R}^n$ such that for all x, $Q^x[\tau_G = 0]$ (resp. $Q^x[\tau_G < \infty]$) is 0; i.e. for all starting points x, X_t does not hit G immediately (resp. never hits G). Moreover, if $U \subset \Omega$ is open and I is the set of irregular points of U, then I is semipolar. With this, we get uniqueness for the generalized Dirichlet problem!

Proposition 3.5. *Suppose $\varphi \in C^0(\partial\Omega)$ is bounded, and $u \in C^2(\Omega)$ is bounded with (i) $Lu = 0$ and (ii) $\lim_{x\to y} u(x) = \varphi(y)$ for all regular $y \in \partial\Omega$. Then $u(x) = E^x[\varphi(X_{\tau_\Omega})]$.*

Proof. Let Ω_k, τ_k be as above; by the Dynkin formula 2.6, we know u is X-harmonic and $u(x) = E^x[u(X_{\tau_k})]$ for all $x \in D_k$ and all k. Now, as $k \to \infty$, $X_{\tau_k} \to X_\tau$, so if X_τ is regular, then $u(X_{\tau_k}) \to \varphi(X_\tau)$. By the above, we know the set of irregular points I of $\partial\Omega$ is semipolar, so in particular, it is polar, which means $X_\tau \notin I$ a.s. Q^x. Thus, bounded convergence gives

$$u(x) = \lim E^x[u(X_{\tau_k})] = \lim E^x\left[\varphi(X_\tau)\right].$$

□

Finally, we can solve the generalized Dirichlet problem:

Theorem 3.6 (Generalized Dirichlet problem for uniformly elliptic operators)**.** *Assume L is uniformly elliptic in Ω, and let $\varphi \in C^0(\partial\Omega)$ be bounded. Define*

$$u(x) = E^x\left[\varphi(X_\tau)\right].$$

Then $u \in C^{2+\alpha}(\Omega)$ for all $\alpha < 1$ and u solves the generalized Dirichlet problem.

Proof. Recall from class (and life in general) that if $B \subset\subset \Omega$ is an open ball, $f \in C^0(\partial B)$, then for all $\alpha < 1$, there exists $u \in C^{2+\alpha}(B) \cap C^0(\overline{B})$ with $Lu = 0$ and $u = f$ on ∂B. Moreover, if K is a compact subset of B, by Schauder theory, there exists a constant C only depending on K and L such that

$$\|u\|_{C^{2+\alpha}(K)} \le C\|f\|_{C^0(\partial B)}.$$

Now, by the previous proposition, we know $u(x) = \int_{\partial B} f(y)d\mu_x(y)$, where $d\mu_x(y) = Q^x[X_{\tau_B} \in dy]$ is the harmonic measure on B, so the Schauder estimates give

$$\left|\int f d\mu_{x_1} - \int f d\mu_{x_2}\right| \le C\|f\|_{C^0(\partial B)}|x_1 - x_2|^\alpha$$

for all $x_1, x_2 \in K$. As this holds for any $f \in C^0(\partial B)$, we get that

$$\|\mu_{x_1} - \mu_{x_2}\| \le C|x_1 - x_2|^\alpha,$$

so any bounded measurable function g on ∂B has

$$\widetilde{g}(x) = \int g(y) d\mu_x(y) = E^x[g(X_{\tau_B})]$$

in $C^\alpha(K)$. Applying this to $u(x) = E^x[u(X_{\tau_U})]$, which holds for all $U \subset\subset \Omega$, we conclude $u \in C^\alpha$ for any compact subset of Ω. This allows us to solve, for any ball $B \subset\subset \Omega$, the problem $Lv = 0$ and $v = u$ on ∂B; by uniqueness, this v again has $v(x) = u(x) = E^x[u(X_{\tau_D})]$, and now we can assume $u \in C^{2+\alpha}$ for all compact subsets of Ω, so $u \in C^{2+\alpha}(\Omega)$.

To get the boundary condition, see [Dyn65, Theorem 13.3]: the idea is that by the theory of parabolic differential equations, one can show that $x \mapsto E^x[f(X_t)]$ is continuous for all $t \geq 0$ and all bounded measurable functions f. Under this condition, $E^x[\varphi(X_\tau)] \to \varphi(y)$ as $x \to y$ for all regular $y \in \partial\Omega$ and φ bounded. □

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