Moore: Semi-Simplicial Complexes & Postnikov Systems

- Idea of "semi-simplicial set" dates back to Eilenberg-Zilber, '50 ("complete semi-s.s.")
- By '56, just "semi-simplicial set"
- Since '60, just "simplicial set" - this paper is from '58.

"If you learned about sSet in the low-tech way, that's ok then,
but I'd feel bad to <u>tell</u> you about this w/o any of the tech"

§1: Foundations

Let Δ be the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$
& order-preserving maps.

Maps here gen'd by maps of these form:
into coface $\partial^i : [n-1] \to [n]$, $0 \le i \le n$ "miss i"
& codegen $\sigma^i : [n+1] \to [n]$, $0 \le i \le n$, "double i"

Def. The category of simplicial sets is $sSet := Set^{\Delta^{op}} = \mathrm{Fun}(\Delta^{op}, Set)$

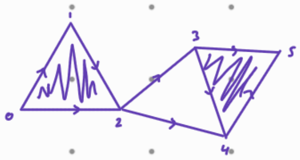
Def. A simp. set $X \in sSet$ is the data of
1) $\forall q \ge 0$, set of q-simplices X_q,
2) face maps $d_i : X_q \to X_{q-1}$ $\quad 0 \le i \le q$
3) degen maps $s_i : X_q \to X_{q+1}$, s.t. (img of degen maps are "degenerate simplices")

$$\begin{cases} d_i d_j = d_{j-1} d_i & i < j \\ s_i s_j = s_{j+1} s_i & i \le j \\ d_j s_j = d_{j+1} s_j = id, & \\ d_i s_j = s_{j-1} d_i & i < j \\ d_i s_j = s_j d_{i-1} & i > j+1 \end{cases}$$

skip

Also, a map of ssets is a morphism of functors.

Cut **Ex.**

X = 

oh, so colim of std simplices? CoYoneda!

Constr.

For $n \ge 0$, let $\Delta^n \in Top$ be <u>topological n-simplex</u>:

$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \le t_i \le 1, \ \Sigma t_i = 1\}$

There are "face" maps $\partial_i : \Delta^{n-1} \to \Delta^n$, $(t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$

& "degen" maps $\sigma_i : \Delta^{n+1} \to \Delta^n$, $(t_0, \ldots, t_{n+1}) \longmapsto (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, \ldots, t_{n+1})$

This gives a "cosimplicial object" $\Delta \xrightarrow{\Delta^\bullet} Top$.

by formal nonsense, we can turn cosimplicial objects in $\mathcal{C}$ into functors from $\mathcal{C} \to sSet$.

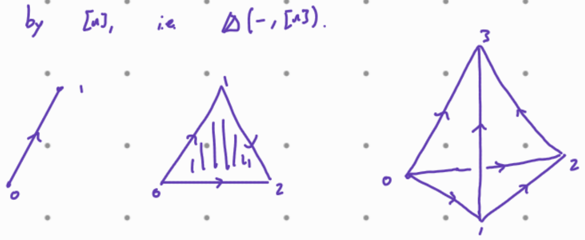
Payoff: For $X \in Top$, define $S(X) \in sSet$ as

$Top(\Delta^\bullet, X)$ "singular cplx of X"

$S(X) = \Delta^{op} \to Top^{op} \to Set$

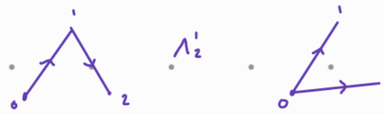
i.e. $S(X)_n = \{\Delta^n \to X\}$, & structure maps are pullback by those of $\Delta^\bullet$

Def. The std. n-simplex $\Delta[n]$ is the sSet represented by $[n]$, i.e. $\Delta(-,[n])$. ($\Delta[-]$ a cosimplicial obj.)



bdry $\partial\Delta[n] := \{$gen'd by $(n-1)$ faces of $\Delta[n]\}$

The k-horn $\Lambda^n_k \subseteq \Delta[n]$ is the bdry minus the k^{th} codim 1 face. $(0 \le k \le n)$



Rmk. By Yoneda, $sSet(\Delta[n], X) = X_n$.

Important Def.

A sSet X is a Kan complex if the following extension condition holds:

$$\begin{tikzcd} \Lambda^k_n \arrow[r, "x"] \arrow[d] & X \\ \Delta[n] \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Explicitly, if given $x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_n \in X_{n-1}$ s.t.

$\forall i<j$ $i,j \neq k$, $d_i x_j = d_{j-1} x_i$, $\exists x \in X_n$ s.t. $d_i x = x_i$.

Thm $S(X)$ is always a Kan cplx. (retracts) + Non example: $\Delta[n]$! paths not invertible

$\Downarrow$

Kan condition asks that sSet is very 'rich' & "full"

MORAL: theory of Kan cplxs is "equivalent" to theory of top. spaces.

In a sense, these are all the examples; $|S(X)| \to X$ is always a W.E.,

but $X \to S(|X|)$ is fibrant replacement — also, $Top \simeq \infty Grpd \simeq Kan\,Cplx$ ($\simeq$ Anima)

Def. For $X, Y \in sSet$,

$(X \times Y)_n = X_n \times Y_n$, str. maps diagonally,

$Map(X,Y)_n = \{X \times \Delta[n] \to Y\}$, str. maps act on $\Delta[n]$ e.g. $d_i f = f \circ (1 \times \delta^i)$

For subcplxs $A \subseteq X$, $B \subseteq Y$, then

$Map((X,A),(Y,B))_n = \{(X \times \Delta^n, A \times \Delta^n) \to (Y,B)\}$

- If Y, B Kan, then so is $(Y,B)^{(X,A)}$!

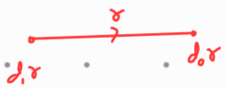
• Also, sSet is cocomplete & complete.

§2: Simplicial Homotopy Theory

• "Points" of X are $X_0 \simeq \{\Delta[0] \to X\}$ (by Yoneda)

• "Paths" are $X_1 \simeq \{\Delta[1] \to X\}$

talk abt htpy groups & their operations



• 2 pts in same "path component" if $\exists\ \gamma \in X_1$ s.t. $d_1\gamma = a$, $d_0\gamma = b$.

Prop. ‖ If X is Kan, then $a \sim b$ is equiv. rel.

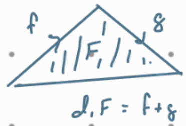
Def. ‖ If X Kan, $\underline{\pi_0 X} := X_0/\sim$, $a \sim b$ iff $\exists\ \gamma \in X_1$ s.t.

• Two maps $f, g : X \to Y$ are $\underline{\text{htpic}}$ if $[f] = [g]$ in $\pi_0(Y^X)$; i.e. $\exists\ F : (X, A) \times \Delta^1 \to (Y, B)$ s.t. $d^1F = f$, $d^0F = g$. (need (Y, B) Kan)

Draw diagram instead.

$$\begin{tikzcd} X \times \Delta^{\{0\}} \arrow[rr, "f"] & & Y \\ X \times \Delta^1 \arrow[u] \arrow[d] \arrow[urr, "F"] & & \\ X \times \Delta^{\{1\}} \arrow[uurr, "g"'] & & \end{tikzcd}$$

Def. | Let X be Kan, $x \in X_0$ giving rise to "bpt" in X_n via $s_0^n x$.
Define
$$\pi_n(X, x) = \pi_0\, \mathrm{Maps}\big((\Delta[n], \partial\Delta[n]) \to (X, x)\big)$$
For $f, g \in \pi_n(X, x)$, let $F : \Delta[n+1] \to X$ be s.t.
$d_{n+1}F = f$, $d_{n-1}F = g$, $d_i F = x$ $\forall\ i < n-1$.
Then define $[f] + [g] = [d_n F] \in \pi_n(X, x)$.



Prop. | $\pi_n(X, x)$ is a group for $n \geq 1$,
& abelian for $n \geq 2$.

$\pi_n(\mathrm{Sing}(X)) \simeq \pi_n(X)$

Functoriality:

1) $(X, x) \xrightarrow{f} (Y, y) \xrightarrow{g} (Z, z)$

$\rightsquigarrow$

$\pi_n(X, x) \xrightarrow{f_*} \pi_n(Y, y) \xrightarrow{g_*} \pi_n(Z, z)$
$= (gf)_*$

& $(id)_* = id$

2) if $f \sim g \in \pi_0\big((Y, y)^{(X, x)}\big)$, then $f_* = g_*$

3) if $f : (X, x) \to (y, y)$, then $f_* = 0$.

To skip

1) "minimality"

2) "twisted products"

Thm. ‖ Fundamental weakness

Thm If Everything works.

§3: Kan Fibrations & Postnikov Sections

∇ Moore calls them "Fiber spaces"

Def $\begin{matrix} E \\ \downarrow p \\ B \end{matrix}$ is a Kan fibration if the following lifting condition holds: think "Serre fibration"

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "{(y_0, y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n)}"] \arrow[d, hook] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "x"'] \arrow[ur, dashed, "\exists y"] & B \end{tikzcd}$$

(explicit?)

$\partial_i y_j = d_{j-1} y_i.$

$p(y_i) = d_i x$

$b \in B_0$, subcplx $p^{-1}(b) \subseteq E$ is F - fiber over B

$$\begin{matrix} F \longrightarrow & E \\ & \downarrow p \\ & B \end{matrix}$$

i.e. $E^{\Delta[n]} \longrightarrow E^{\Lambda^k_n} \times_{B^{\Lambda^k_n}} B^{\Delta[n]}$ is surj.

Prop. F is Kan, & E Kan iff B is.

Constr. LES of htpy!

(1) $\forall q \geq 2$, let $\partial : \pi_q(B, b) \longrightarrow \pi_{q-1}(F, a)$

$\alpha \in \pi_n B$ rep'd by $x \in B_q$ st. $d_i x = s_0^{q-1} b \ \ \forall i$

Using Kan lifting, can lift x to $y \in E_q$ w/ $p(y) = x$

& $d_i y = s_0^{q-1} a \ \ \forall i > 0$. So $d_0 y \in F_{q-1}$.

$\partial : \alpha \mapsto [d_0 y]$

(2) Check this is hom.

(3) The following is exact:

$$\cdots \to \pi_q(F, a) \xrightarrow{i_*} \pi_q(E, a) \xrightarrow{p_*} \pi_q(B, b) \xrightarrow{\partial} \pi_{q-1}(F, a) \to \cdots$$

Constr: Postnikov Sections:

Goal: decompose Kan cplx X into htpically simpler things.

1) $\forall n \geq 0$, define $X_{\leq n}$ as $X/\sim$,

w/ $x \sim x'$ if $x|_{sk_n X} = x'|_{sk_n X}$ Cells above dim n

$\Rightarrow \pi_{>n} X_{\leq n} = 0$, $\pi_{\leq n} X_{\leq n} = \pi_{\leq n} X$.

& $X \longrightarrow X_{\leq n}$ is fibration.

2) Following pictures:

a) fiber $\vdots$ $\downarrow$

b) under ...

are EM cplxs.

$$\begin{array}{ccccc} & & X_{\le 3} & & \\ & & \downarrow & & \\ K(\pi_2 X, 2) & \longrightarrow & X_{\le 2} & \xrightarrow{k'} & K(\pi_3, 4) \\ & & \downarrow & & \\ K(\pi_1 X, 1) & \longrightarrow & X_{\le 1} & \xrightarrow{k'} & K(\pi_2, 3) \\ & & \downarrow & & \\ & & X_{\le 0} \simeq \Delta^0 & & \end{array}$$

condition (on $\pi_1 X \curvearrowright \pi_n X$)

the are lays of

$\Rightarrow$ "k-invariants"

$$k^{n+2} : X \longrightarrow K(\pi_{n+1} X, n+2)$$

$$\in H^{n+2}(X ; \pi_{n+1})$$

sSet has natural "cell structure"

Moral: these control how "twisted" our space is.

→ Moore makes explicit models of the k-invariants through "twisted Cartan products", which are like simplicial G-bundles.

§4 - Intro to Dold-Kan

Finally, I want to show basically the most important construction you can do w/ simplicial abelian groups:

Def. || $sAb = \mathrm{Fun}(\Delta^{op}, Ab)$

Ex. $\mathbb{Z}[\mathrm{Sing}(X)]$ for any $X \in Top$ — singular chain cplx!

Thm || There is an equiv. $sAb \simeq Ch_{\ge 0}(\mathbb{Z})$

Constr

$$sAb \xrightarrow{N} Ch_+ \mathbb{Z}$$
$$X \longmapsto NX \quad \text{"Moore cplx"}$$

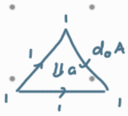
$$(NX)_n = \bigcap_{i=1}^{n} \ker d_i : X_n \to X_{n-1}, \qquad d : NX_n \to NX_{n-1} = d_0$$

Alt, $CX_n = X_n$, $d = \sum (-1)^n d_n$

$CX \simeq NX \in \mathcal{D}(\mathbb{Z})$

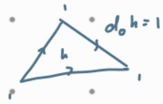
k-chains: k-disks in X capped in single fn.

$1 \xrightarrow{a} d_0 a \in NA_1$ (1-disk, i.e. pt)



$\in NA_2$ (2-disk)

$h \in Z_2(NA)$ if



(closed to 2-sphere)

$\Rightarrow$ $\pi_n X \cong H_n(NX)$ ← either def. of π_n or direct comp.

Get functor back $\Gamma : Ch_+ \mathbb{Z} \to sAb$,

via nerve assoc to cosimplicial obj.

$$\mathbb{Z}^{\bullet}[-] : \Delta \to Ch_+ \mathbb{Z}$$

$$[k] \longmapsto N(\mathbb{Z}^{\Delta[k]})$$

↑ free simp. ab. grp on $\Delta[n]$

$\therefore\ \Gamma(V)_k = \mathrm{Hom}_{Ch_{\geq}\mathbb{Z}}(N(\mathbb{Z}^{\Delta^k}), V)$

explicitly, $\Gamma(V)_n = \bigoplus_{[n] \twoheadrightarrow [k]} V_k$

& maps are some funky thing.

Moral: This equiv. is as desired as you want; on one hand is htpy thy of

sAb ($\simeq sSet$) & on other is $\mathcal{D}(\mathbb{Z})_{\geq 0}$

↳ gives you EM objs in sAb.

This is HA 1.2.4

Dold-Thom; $\pi_* SP(X) \simeq H_* X$ naturally in X

$\Rightarrow$ (Hatcher 4K.7) Every path-ctd, commutative $\mathbb{E}_1$-space (assoc, strictly unital H-space) has weak htpy type of generalized EM space.

& In Moore, this is shown for $X \in sAb$:

$X \simeq \bigoplus K(\pi_n X, n)$,

b/c (Dold-K), or explicitly b/c k-invariants are all zero!

n.b. — ask for question