POINCARÉ INEQUALITIES AND THE GEOMETRY OF POINCARÉ CONSTANTS

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Contents

1. The Poincaré inquality and Riemann surface theory

Let X be a compact, connected Riemann surface, with a fixed area form ω. For a (smooth) function f on X, we can define the L^2 norm and Dirichlet (semi)norm as

$$\|f\|_{L^2} := \left(\int_X |f|^2\omega\right)^{1/2}, \quad \|f\|_D := \left(\int_X |Df|^2\omega\right)^{1/2},$$

where Df is the gradient of f. The Poincaré inequality relates these two quantities:

Proposition 1.1. *There exists a constant C depending on X and ω such that for all $f \in \Omega^0(X;\mathbf{R})$,*

$$\|f\|_{L^2} \leq C\|f\|_D \quad \textit{if} \quad \int_X f\omega = 0.$$

In class, we proved this by first examining this inequality on the sphere, generalizing to balls with various area forms via a reflection principle and patching these together for an arbitrary compact connected Riemann surface. Moreover, this is a key non-formal input in establishing the "main theorem" of Riemann surfaces, which we leveraged to great effect in constructing the theory of divisors.

This is one indication that this kind of inequality, bounding the L^2 norm of a function by the L^2 norm of its gradient, has wide-ranging applications. In fact, more general Poincaré inequalities are of critical importance to a wide range of subfields, and the constants in these inequalities carry significant geometric content. In this expository paper, I'll discuss the importance of the

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Poincaré inequality to the analytic theory of Riemann surfaces, present and discuss proofs of some forms of this inequality, and give a brief overview of the connection between Poincaré constants and the geometry of their domains.

These is my final paperer for Math 213B, a Spring 2024 course on Riemann surfaces taught by Peter Kronheimer. All mistakes are my own; please reach out to me if you spot anything!

1.1. **Relation to the Main Theorem.** Recall from lecture the "Main Theorem" of Riemann surfaces was about solving the integral Poisson equation in differential forms. Much of the later theory (Riemann-Roch, etc.) stemmed from this result:

Theorem 1.2. *Let ρ be a 2-form on X. Then there exists a solution ϕ to $\Delta\phi = \rho$ iff the integral of ρ on X is 0, and the solution is unique up to constants.*

Here Δ is the geometers' Laplacian,[1] $\Delta = 2i\bar{\partial}\partial$. Of course, if $\rho = \Delta\phi$, then $\int_X \rho = 0$ by Stokes' theorem (X is naturally a closed 2-manifold). Moreover, the only harmonic functions on X are constant, by the maximum principle. Thus, the *hard* part of this main theorem is the converse direction. Let us recap the proof from lecture, to see the Poincaré inequality in action.

Proof sketch. First, let $H = \Omega^0(X;\mathbf{R})/\mathbf{R}$ be the space of smooth functions on X, modulo constants, and ρ a fixed 2-form with $\int_X \rho = 0$. For $\phi \in H$, define

$$L(\phi) = \langle \phi, \phi\rangle_D - 2\int_X \phi\rho.$$

The key observation is that L is a sort of Euler-Lagrange functional for the equation $\Delta\phi = \rho$; that is, if ϕ^* minimizes L, then for any $\psi \in H$, we have

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}[L(\phi^* + t\psi)|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\langle \phi^*, \phi^*\rangle_D + 2t\langle \phi^*, \psi\rangle_D + t^2\langle\psi,\psi\rangle_D - 2\int_X (\phi^* + t\psi)\rho\right|_{t=0} \\ &= 2\langle \phi^*, \psi\rangle_D - 2\int_X \psi\rho \\ \implies 0 &= \int_X \psi(\Delta\phi^* - \rho). \end{aligned}$$

As this holds for all ψ, this implies $\Delta\phi^* = \rho$, by a "fundamental lemma of variational calculus" in our geometric setting. Thus, minimizers of L correspond to solutions of the Poisson equation, and it suffices to show L achieves this minimum on H:

1. First, L is bounded below: writing $\rho = R\omega$, Cauchy-Schwarz and the Poincaré inequality above give
$$\int_X \phi R\omega \le \|R\|_{L^2}\|\phi\|_{L^2} \le C'\|\phi\|_D,$$
where C' is a constant depending on the Poincaré constant and ρ, which is fixed in this context. Here we use the observation that for $\phi \in H$ nonzero, $\int \phi\omega = 0$, as for fixed ω, we can identify H with smooth functions with integral 0 (by subtracting off the total integral). Thus,
$$L(\phi) \ge \|\phi\|_D^2 - 2C'\|\phi\|_D = (\|\phi\|_D - C')^2 - 4(C')^2 \ge -4(C')^2,$$
so the infimum $\inf_H L$ exists in $\mathbf{R}$.

[1] As much as it breaks my inner analyst's heart.

2. Next, we can show by a direct computation that a minimizing sequence $\phi_i \in H$ with $L(\phi_i) \to \inf_H L$ is Cauchy with respect to the Dirichlet norm.
3. We can now complete H, which is an inner product space under the Dirichlet norm (that this is indeed a norm and not just a seminorm also uses the Poincaré inequality). Let $\widehat{H}$ be this completion, consisting of equivalence classes of Cauchy sequences.
4. L is continuous on H by the Poincaré inequality, so extends to the completion $\widehat{H}$; moreover, by the previous fact, a minimizing (hence Cauchy) sequence for L in H has a tautological limit ϕ in $\widehat{H}$, which by continuity has $L(\phi) = \inf_H L = \inf_{\widehat{H}} L$.
5. We now have a minimizing Cauchy sequence $\phi_i \to \phi$, which again by the Poincaré inequality is Cauchy in L^2. By completeness of L^2, we can find $\Phi \in L^2$ with ϕ_i converging to Φ in L^2. A simple computation now gives
$$\int_X \Phi \Delta \psi - \int_X \rho \psi = 0$$
for all $\psi \in H$. Thus, Φ (which can be thought of as a representation of $\phi \in \widehat{H}$) is a weak solution to the Poisson equation.
6. If Φ was smooth, we would be done. However, we are still okay, as by Weyl's lemma (which we proved in the course of the homework), a weak solution to the Poisson equation is a.e. given by a smooth function, thus completing the proof.

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A few comments are in order: most of this proof is fairly straightforward functional analysis. The only two non-formal inputs are the Poincaré inequality (which is used multiple times) and Weyl's lemma (which is used in constructing a smooth solution). Of these, the latter is a typical result of elliptic regularity theory (yippee!), while the former has much more geometric content, as we'll soon see.

2. Three proofs of the general Poincaré inequality

With this motivation in hand, we can appreciate how useful the Poincaré inequality is, but how might we prove it? Here, we will consider three versions of this inequality on subsets of $\mathbf{R}^n$, with three corresponding proofs, examining how they trade off generality (in the domain) with stronger constants. In the next section we'll look in passing at estimates on manifolds, but working locally (as we did in class), assuming we are in $\mathbf{R}^n$ is usually sufficient.

For us, a domain $\Omega \subset \mathbf{R}^n$ is a connected, nonempty open set. Any property on the regularity of the boundary (smooth, C^1, Lipschitz) should be understand as asking for the same regularity on $\partial\Omega$ as a manifold. All functions considered below are assumed to be (Lebesgue) measurable.

For this section, fix p between $1 \le p < \infty$. Recall that a function f on Ω has L^p norm
$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \, \mathrm{d}x \right)^{1/p},$$
and $f \in L^p(\Omega)$ if $\|f\|_{L^p(\Omega)} < \infty$. The L^p spaces are complete, and contains smooth functions with compact support (denoted as C_c^∞) as a dense subspace. Poincaré inequalities involve comparing the L^p norm of a function with the L^p norm of its derivative, so we need to restrict to functions for which the latter phrase makes sense. The following definition is standard:

Definition 2.1. Suppose $u, v \in L^1_{\text{loc}}(\Omega)$ (meaning L^1 on compact subsets), and $i = 1, \dots, n$. We say v is the *ith weak partial derivative* of u, writing $D_i u = v$, if

$$\int_\Omega u D_i \phi \, \mathrm{d}x = -\int_\Omega v\phi \, \mathrm{d}x \text{ for all } \phi \in C_c^\infty(\Omega)$$

Note that if u was C^1, this would hold via integration by parts with v being the "true" partial derivative. It is a fact that weak derivatives are uniquely defined a.e., if they exist. (By contrast, *distributional* derivatives always exist, and are defined by a similar property of checking against test functions. However, weak derivatives are at least functions).

Definition 2.2. The *Sobolev space* $W^{1,p}(\Omega)$ consists of all $L^p(\Omega)$ functions u such that for each $i = 1, \dots, n$, the weak derivative $D_i u$ exists and is $L^p(\Omega)$. The Sobolev norm is given by

$$\|u\|_{W^{1,p}} = \left(\int_\Omega |u|^p \, \mathrm{d}x + \sum_{i=1}^n \int_U |D_i u|^p \, \mathrm{d}x \right)^{1/p}.$$

$C_c^\infty(\Omega)$ is naturally a subspace of $W^{1,p}(\Omega)$, and we define

$$W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}$$

to be the space of Sobolev functions which are "compactly supported."

Sobolev spaces are natural choices for function spaces in which we want to do differential analysis or solve PDE, but without restricting all the way to smooth functions. Note that the choice of norm above is one of many equivalent choices.

2.1. **A compactness approach.** Our first version of the Poincaré inequality comes from [Eva10], and uses the following result, which we saw on the homework.

Theorem 2.3 (Rellich-Kondrachov Compactness)**.** *Suppose $\Omega \subset \mathbf{R}^n$ is a bounded domain and $\partial\Omega$ is C^1. Then*

$$W^{1,p}(\Omega) \subset\subset L^p(\Omega).$$

That is, $\|\cdot\|_{L^p} \leq C\|\cdot\|_{W^{1,p}}$ for a constant C and any bounded sequence in $W^{1,p}(\Omega)$ has a convergent subsequence in $L^p(\Omega)$.

See [Eva10, §5.7] for a proof. The comparison of norms is trivial in our case, but in general, when the exponents of these two spaces are allowed to be different, this result is known as the Gagliardo-Nirenberg-Sobolev inequality. For the convergent subsequence part, one can try a hands-on diagonalization argument, or first mollify and reduce to considering sequences of smooth functions and apply Arzela-Ascoli. Nothing about the nature of Ω is used here, except for boundedness, the consequence of which we'll see in the below estimate.

Theorem 2.4. *Let Ω be a bounded domain in $\mathbf{R}^n$ with C^1 boundary. Then there exists a constant $C = C(n, p, U)$ such that*

$$\left\| u - \fint_\Omega u \, \mathrm{d}x \right\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$$

Proof. I learned this proof from [Eva10, §5.8.1]. Write

$$\mathbf{E}_\Omega[u] := \fint_\Omega u \, \mathrm{d}x = \frac{1}{|\Omega|} \int_\Omega u \, \mathrm{d}x$$

for the average of u over Ω. We proceed by contradiction: if no such constant existed, then for each $k \geq 1$, we could find $u_k \in W^{1,p}$ such that

$$\|u_k - \mathbf{E}_\Omega[u_k]\|_{L^p(\Omega)} > k\|D_k u\|_{L^p(\Omega)}.$$

Define v_k by centering $u_k - \mathbf{E}_\Omega[u_k]$ and rescaling so that $\|v_k\|_{L^p} = 1$ and $\mathbf{E}_\Omega[v_k] = 0$:

$$v_k(x) = \frac{u_k(x) - \mathbf{E}_\Omega[u_k]}{\|u_k - \mathbf{E}_\Omega[u_k]\|_{L^p(\Omega)}}, \quad Dv_k = \frac{Du_k}{\|u_k - \mathbf{E}_\Omega[u_k]\|_{L^p(\Omega)}}.$$

That is, $\|Dv_k\| < \frac{1}{k}$. In particular, $\{v_k\}$ is bounded in $W^{1,p}(\Omega)$, so by Theorem 2.3, there exists $w \in L^p(\Omega)$ with a subsequence $\{w_k\} \subset \{v_k\}$ converging to w in L^p. Thus, this w has $\mathbf{E}_\Omega[w] = 0$ and $\|w\|_{L^p(\Omega)} = 1$. However, as the derivatives of w_k are converging to 0, this w should also have zero (weak) derivative: indeed, for any test $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega w D_i\phi \,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega w_k D_i\phi \,\mathrm{d}x = -\lim_{k\to\infty} \int_\Omega (D_i w_k)\phi \,\mathrm{d}x = 0.$$

Thus, $Dw = 0$ a.e., and since Ω is connected, w is constant. However, $\mathbf{E}_\Omega[w] = 0$ implies $w = 0$, but $\|w\|_{L^p(\Omega)} = 1$, a contradiction. //

Here, we get no control on the constant; it shows up because our (very general and nonetheless powerful) compactness theorem declares it must show up. While a useful and, in my opinion, fairly quick proof, it lacks some of the intuition of the next two.

Also, it is a similar fact that $W_0^{1,p}(\Omega) \subset\subset L^p(\Omega)$ for such bounded domains, and this same proof adapts readily to a Poincaré inequality for such zero boundary functions. There the contradiction is (roughly) that the w so constructed is contained in $W_0^{1,p}(\Omega)$, so should be 0 on the boundary, but it is also a constant with integral 1, which is impossible. [Thi23].

It is worthwhile to note, however, that rescaling gives a version of domain dependence:

Corollary 2.5. *There exists a constant $C = C(n,p)$ such that for $u \in W^{1,p}(B_r(x))$,*

$$\left\| u - \mathbf{E}_{B_r(x)}\, u \right\|_{L^p(B_r(x))} \leq Cr\|Du\|_{L^p(B_r(x))},$$

where $B_r(x)$ denotes the radius r open ball around $x \in \mathbf{R}^n$.

Proof. Pick C in Theorem 2.4 corresponding to $\Omega = B_1(0)$ being the centered unit ball. Then, reparametrize

$$v(y) = u(x + ry), \quad y \in B_1(0).$$

It is clear that $\left\| u - \mathbf{E}_{B_r(x)}\, u \right\|_{L^p(B_r(x))} = \left\| v - \mathbf{E}_{B_1(0)}\, v \right\|_{L^p(B_1(0))}$, and similarly,

$$\int_{B_1(0)} |Dv|^p \,\mathrm{d}y = \int_{B_1(0)} r^p |Du(x+ry)|^p \,\mathrm{d}y \implies \|Dv\|_{L^p(B_1(0))} = r\|Du\|_{L^p(B_r(x))}.$$

Applying Theorem 2.4 to v and using these two inequalities gives the desired estimate. //

In general, the existence of Sobolev inequalities with differing parameters can be ruled out by such scaling arguments, but we won't touch more on this idea: see the notes [Joh20].

2.2. **A direct approach.** he next version I will present uses the domain more directly, as well as a direct computation, to give a more interpretable constant.

In this and the next section, let $q = \frac{p}{p-1}$ be Hölder conjugate to p.

Theorem 2.6. *Let Ω be a finite width domain, i.e. it lies between two parallel hyperplanes in $\mathbf{R}^n$ with width d. Then for all $u \in W_0^{1,p}(\Omega)$, we have*

$$\int_\Omega |u|^p \, dx \leq \frac{d^p}{p} \int_\Omega |Du|^p \, dx.$$

Proof. I learned this proof from [Leo17, Theorem 13.19]. By density (vis-a-vis definition) it suffices to assume $u \in C_c^\infty(\Omega)$, and by rotating/translating, assume without loss of generality Ω lies between the hyperplanes $x_n = 0$ and $x_n = d$. We can compute

$$\begin{aligned} |u(x', x_n)| = |u(x', x_n) - u(x', 0)| &= \left| \int_0^{x_n} D_n u(x', t) \, dt \right| \\ &\leq \int_0^d \mathbf{1}_{t \leq x_n} |D_n u(x', t)| \, dt \leq x_n^{1/q} \left(\int_0^d |D_n u(x', t)|^p \, dt \right)^{1/p}, \end{aligned}$$

by the fundamental theorem of calculus and Hölder's inequality. Extending u by 0 on $\mathbf{R}^n \setminus \Omega$, we can estimate the LHS using Fubini-Tonelli, and that $|D_n u| \leq |Du|$:

$$\begin{aligned} \int_\Omega |u|^p \, dx &= \int_{\mathbf{R}^{n-1}} \int_0^d |u(x', x_n)|^p \, dx_n \, dx' \\ &\leq \int_{\mathbf{R}^{n-1}} \int_0^d x_n^{p/q} \int_0^d |D_n u(x', t)|^p \, dt \, dx_n \, dx' \\ &= \int_0^d x_n^{p-1} \, dx_n \cdot \int_{\mathbf{R}^{n-1}} \int_0^d |D_n u(x', t)|^p \, dt \, dx' \\ &= \frac{d^p}{p} \int_\Omega |D_n u|^p \, dx \leq \frac{d^p}{p} \int_\Omega |Du|^p \, dx. \end{aligned}$$

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In many sources, this theorem is stated with the stronger assumption that Ω is bounded; here we only need that Ω is bounded in one direction, as the key step is being able to control u by integrating the gradient in that direction. Of course, bounding $|D_n u| \leq |Du|$ might lose a lot in general, but it can be tight if, say, u is constant in all other directions.

2.3. **A volumetric approach.** Jost's book [Jos13] has yet another version of the Poincaré inequality, equivalent at least in spirit to that of Leoni. However, the proof makes critical use of the following volumetric estimate. Jost's treatment of the subject focuses on the special (and Hilbert space theoretic) case of $W^{1,2}$, but I have adapted the proof to arbitrary p. Let ω_n be the measure of the unit ball in $\mathbf{R}^n$.

Lemma 2.7. *Let Ω be a domain with finite (Lebesgue) measure, $f \in L^1(\Omega)$, $0 < a \leq 1$. Then*

$$\left\| \int_\Omega |x - y|^{n(a-1)} f(y) \, dy \right\|_{L^p(\Omega)} \leq \frac{1}{a} \omega_n^{1-a} |\Omega|^a \|f\|_{L^p(\Omega)}.$$

Proof. Pick $R \geq 0$ so that $|\Omega| = |B_R(x)| = \omega_n R^n$. In particular,

$$|\Omega \setminus (\Omega \cap B_R(x))| = |B_R(x) \setminus (\Omega \cap B_R(x))|,$$

and for $y \notin B_R(x)$, $|x-y|^{n(a-1)} \leq R^{n(a-1)}$, and vice versa for $y \in B_R(x)$. Thus,

$$\int_\Omega |x-y|^{n(a-1)}\,\mathrm{d}y \leq \int_{B_R(x)} |x-y|^{n(a-1)}\,\mathrm{d}y = \frac{1}{a}\omega_n R^{na} = \frac{1}{a}\omega_n^{1-a}|\Omega|^a.$$

as on $\Omega \setminus (\Omega \cap B_R(x))$, these assumptions let us perform the following bit of clever trickery:

$$\begin{aligned}
\int_{\Omega\setminus(\Omega\cap B_R(x))} |x-y|^{n(a-1)}\,\mathrm{d}y &\leq \int_{\Omega\setminus(\Omega\cap B_R(x))} R^{n(a-1)}\,\mathrm{d}y \\
&= |\Omega \setminus (\Omega \cap B_R(x))| R^{n(a-1)} \\
&= |B_R(x) \setminus (\Omega \cap B_R(x))| R^{n(a-1)} \\
&= \int_{B_R(x)\setminus(\Omega\cap B_R(x))} R^{n(a-1)}\,\mathrm{d}y \\
&\leq \int_{B_R(x)\setminus(\Omega\cap B_R(x))} |x-y|^{n(a-1)}\,\mathrm{d}y.
\end{aligned}$$

The remaining computation follows by radial integration, introducing the appropriate volume elements. Therefore, writing

$$|x-y|^{n(a-1)}|f(y)| = |x-y|^{n(a-1)/q} \cdot |x-y|^{n(a-1)/p}|f(y)|,$$

Hölder's inequality gives

$$\begin{aligned}
\left|\int_\Omega |x-y|^{n(a-1)} f(y)\,\mathrm{d}y\right| &\leq \int_\Omega |x-y|^{n(a-1)}|f(y)|\,\mathrm{d}y \\
&\leq \left(\int_\Omega |x-y|^{n(a-1)}\,\mathrm{d}y\right)^{1/q}\left(\int_\Omega |x-y|^{n(a-1)}|f(y)|^p\,\mathrm{d}y\right)^{1/p}
\end{aligned}$$

Integrating this inequality over $x \in \Omega$ gives

$$\begin{aligned}
\left\|\int_\Omega |x-y|^{n(a-1)} f(y)\,\mathrm{d}y\right\|_{L^p(\Omega)}^p &\leq \left(\frac{1}{a}\omega_n^{1-a}|\Omega|^a\right)^{p/q} \int_\Omega\int_\Omega |x-y|^{n(a-1)}|f(y)|^p\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \left(\frac{1}{a}\omega_n^{1-a}|\Omega|^a\right)^{p-1} \int_\Omega\int_\Omega |x-y|^{n(a-1)}|f(y)|^p\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \left(\frac{1}{a}\omega_n^{1-a}|\Omega|^a\right)^{p-1} \int_\Omega\left(\frac{1}{a}\omega_n^{1-a}|\Omega|^a\right)|f(y)|^p\,\mathrm{d}y \\
&\leq \left(\frac{1}{a}\omega_n^{1-a}|\Omega|^a\right)^{p} \|f\|_{L^p(\Omega)}^p \\
\implies \left\|\int_\Omega |x-y|^{n(a-1)} f(y)\,\mathrm{d}y\right\|_{L^p(\Omega)} &\leq \left(\frac{1}{a}\omega_n^{1-a}|\Omega|^a\right)\|f\|_{L^p(\Omega)}.
\end{aligned}$$

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So what just happened there? The key step is the volumetric rearrangement, which manages to be independent of the point x and control the integrals of these sharp spikes; the remaining steps are fairly rote. With this in hand, we get a new version of the Poincaré inequality, relating to the measure of Ω instead of its diameter.

Theorem 2.8. *Let Ω be a domain with finite measure. Then for $u \in W_0^{1,p}(\Omega)$, we have*

$$\|u\|_{L^p(\Omega)} \leq \left(\frac{|\Omega|}{\omega_n}\right)^{1/n} \|Du\|_{L^p(\Omega)}.$$

Proof. I learned this proof from [Jos13, Theorem 8.2.2], where it is proved for $p = 2$; the extension to general p is my own. As before, it suffices to assume $u \in C_0^1(\Omega)$ (this is still dense in $W_0^{1,p}(\Omega)$) and extend u by 0 outside of Ω. Now, let $\omega \in S^{n-1}$ be any unit length direction in $\mathbf{R}^n$. Integrating u on the ray $\mathbf{R} \cdot \omega$, we get

$$u(x) = -\int_0^\infty D_r u(x + r\omega)\,\mathrm{d}r.$$

Then, integrating over ω gives

$$\begin{aligned} u(x) &= -\int_0^\infty \fint_{|\omega|=1} D_r u(x + r\omega)\,\mathrm{d}\omega\,\mathrm{d}r \\ &= -\frac{1}{n\omega_n}\int_0^\infty \int_{|\omega|=1} D_r u(x + r\omega)\,\mathrm{d}\omega\,\mathrm{d}r \\ &= -\frac{1}{n\omega_n}\int_0^\infty \int_{y\in\delta B_r(x)} \frac{1}{r^{n-1}}\frac{\partial u}{\partial \nu}(y)\,\mathrm{d}\sigma(z)\,\mathrm{d}r \\ &= -\frac{1}{n\omega_n}\int_\Omega \frac{1}{|x-z|^{n-1}}\sum_{i=1}^n D_i u(z)\cdot\frac{x_i - z_i}{|x-z|}\,\mathrm{d}z \\ \implies |u(x)| &\leq \frac{1}{n\omega_n}\int_\Omega \frac{1}{|x-z|^{n-1}}|Du(z)|\,\mathrm{d}z. \end{aligned}$$

The key steps here are few changes of variables, our explicit knowledge of the normal vector, and hence normal derivative, on the ball, and the estimate

$$\left|\sum_{i=1}^n D_i u(z)\cdot\frac{x_i - z_i}{|x-z|}\right| \leq \sum_{i=1}^n |D_i u(z)|\cdot\frac{|x_i - z_i|}{|x-z|} \leq \left(\max_{1\leq i\leq n}\frac{|x_i - z_i|}{|x-z|}\right)\sum_{i=1}^n |D_i u(z)| \leq 1\cdot|Du(z)|.$$

From here, simply apply Lemma 2.7 with $f = |Du|$ and $\mu = 1/n$, giving

$$\|u\|_{L^p(\Omega)} \leq \frac{1}{n\omega_n}\cdot\frac{1}{\mu}\omega_n^{1-1/n}|\Omega|^{1/n}\|Du\|_{L^p(\Omega)} = \left(\frac{|\Omega|}{\omega_n}\right)^{1/n}\|Du\|_{L^p(\Omega)}.$$

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This estimate is fairly surprising to me; many sources claim that Poincaré inequalities *only* work when the domain is bounded in at least 1 direction, but there are many finite measure domains which are unbounded. Most counterexamples involve working on unbounded domains with infinite measure, but the proof above wouldn't apply in such a context nonetheless. It's a curious thing to think about, but in any case, which of Theorem 2.6 and Theorem 2.8 would give better constants depends heavily on the geometry of the domain.

3. The geometry of Poincaré constants

In this section, I'll share some results connecting the Poincaré constant to more precise geometry of the domain. As we'll see, information flows both ways, with stronger geometric assumptions giving better constants, and with known estimates on these constants providing arguably geometric data.

3.1. L^2 Poincaré constants and spectral theory of the Laplacian.

As we saw in class, L^2 Poincaré constants are related to the lowest nonzero eigenvalue of the Laplacian. In fact, a simple calculation can elucidate this connection.

Proposition 3.1. *Let $w > 0$ be a positive smooth function on a domain Ω with $\Delta w \geq \mu w$ for some $\mu > 0$. Then for any $u \in W_0^{1,2}(\Omega)$,*

$$\int_\Omega u^2 \,\mathrm{d}x \leq \frac{1}{\mu}\int_\Omega |Du|^2 \,\mathrm{d}x.$$

Proof. Recall we are using the geometers' Laplacian. Now let u be any smooth function vanishing on $\partial\Omega$. Then $u^2 Dw/w$ also vanishes on $\partial\Omega$, and has divergence

$$\begin{aligned}\operatorname{divv}\left(u^2\frac{Dw}{w}\right) &= 2u\left\langle Du, \frac{Dw}{w}\right\rangle - u^2\frac{|Dw|^2}{w^2} - u^2\frac{\Delta w}{w} \\ &= -\left|Du - u\frac{Dw}{w}\right|^2 + |Du|^2 - u^2\frac{\Delta w}{w} \\ &=\leq |Du^2| - \mu u^2.\end{aligned}$$

Thus, the divergence theorem gives, upon integrating over Ω, that

$$0 \leq \int_\Omega |Du|^2 - \mu\int_\Omega u^2 \implies \mu\int_\Omega u^2 \leq \int_\Omega |Du|^2.$$

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So far this is just calculus: to see that this bound is achieved requires some spectral theory of the Laplacian. That is, one needs a theorem like the following:

Theorem 3.2. *[Jos13, Theorem 9.5.1] Let $\Omega \subset \mathbf{R}^n$ be a bounded domain. Then the eigenvalue problem $\Delta u = \lambda u$, $u \in W_0^{1,2}(\Omega)$ has countably many eigenvalues*

$$0 < \lambda_1 < \lambda_2 \leq \dots,$$

with $\lambda_m \to \infty$ and an L^2-orthonormal basis of eigenfunctions $u_i \in W_0^{1,2}(\Omega)$ with $\langle Du_i, Du_i\rangle = \lambda_i$.

Recall this led to a sharp Poincaré inequality in the following direct way: given any $u \in W_0^{1,2}(\Omega)$, we can write it uniquely as

$$u = \sum_{i\geq 1} a_i u_i \text{ and } Du = \sum_{i\geq 1} a_i Du_i.$$

Then, we can compute the L^2 norms as

$$\|u\|^2_{L^2(\Omega)} = \sum_{i\geq 1} a_i^2 \text{ and } \|Du\|^2_{L^2(\Omega)} = \sum_{i\geq 1}\lambda_i a_i^2.$$

It is clear that

$$\|Du\|^2_{L^2(\Omega)} = \sum_{i\geq 1}\lambda_i a_i^2 \geq \lambda_1\sum_{i\geq 1} a_i^2 = \lambda_1\|u\|^2_{L^2(\Omega)},$$

giving the inequality

$$\|u\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\lambda_1}}\|Du\|_{L^2(\Omega)},$$

which is of course achieved for $u = u_1$.

Above, we've considered only the case where our functions are 0 on some boundary, but we could have asked for them to have mean 0. One distinguishes these two boundary conditions for Poincaré inequalities by talking about *Dirichlet eigenvalues* (where u is 0 on the boundary) on Neumann eigenvalues (where the normal derivative is 0, or on a closed manifold M, $\int_M u = 0$). In fact, a similar spectral theorem holds for Neumann eigenvalues, and the smallest nonzero such eigenvalues admit the variational characterizations

$$\lambda_1 = \inf \frac{\int_\Omega |Du|^2}{\int_\Omega u^2},$$

where in the Dirichlet case the infimum is taken over zero-boundary functions, and in the Neumann case over mean zero functions. These give corresponding Poincaré inequalities as one expects [CM, §7.1].

To tie back to lecture, our Poincaré inequality on the sphere relied on the spectral theory of Neumann eigenvalues, where constant functions correspond to a nontrivial 0 eigenspace.

In these cases, geometric analysis lets one lower bound these eigenvalues, and hence upper bound the corresponding Poincaré constants. As an example of such a result for readers familiar with some Riemannian geometry, we have the following [CM, Theorem 7.56].

Theorem 3.3 (Lichnerowicz)**.** *Suppose M is a complete Riemannian n-manifold with Ricci curvature lower bounded by $c > 0$. Then the first non-zero Neumann eigenvalue is lowed bounded by $\frac{cn}{n-1}$. In particular, we have*

$$\|u\|_{L^2(M)} \leq \sqrt{\frac{n-1}{cn}}\|Du\|_{L^2(M)}.$$

To avoid getting too deep, I'll refrain from commenting further on this, other than to say that while these functions are usually taken to be smooth, we can still extend them via density.

Thus, much of the literature on L^2 Poincaré constants is really about eigenfunctions of the Laplacian and spectral geometry. For example, a very lower weak bound on this eigenvalue is related to the isoperimetry of the domain Ω [Wei56], although we won't touch on this further. Ergo, if you can understand the spectrum of the manifold Laplacian (e.g. the spherical Laplacian, as we saw in class), you can control the Poincaré constant, whereas an oft-quoted definition of this spectrum comes from its minimax characterization in terms of Poincaré inequalities!

3.2. L^1 **Poincaré constants and isoperimetry.** In the convex setting, i.e., when Ω is a bounded convex domain in $\mathbf{R}^n$ with diameter d, then for $u \in W^{1,p}(\Omega)$ with $\int_\Omega u \, \mathrm{d}x = 0$, we have optimal geometric bounds on the optimal Poincaré constant:

$$C_p = \sup \frac{\|u\|_{L^p(\Omega)}}{\|Du\|_{L^p(\Omega)}}.$$

In particular, for $p = 2$, [PW60] showed the optimal constant is d/π. More recently, for $p = 1$, [AD04] showed the optimal constant is $d/2$. These constants are approached for a parallelepiped with all but one dimension tending to 0, e.g. $\Omega_\epsilon = [0,1] \times [0,\epsilon]^{n-1}$. In both cases, these results rely on first localizing to a 1-dimensional inequality and performing some staightforward analysis. However, I'd like to conclude by reproducing a (very) heuristic argument relating such an L^1 Poincaré inequality to the isoperimetry of the function's level sets [Alg12].

Suppose we have a function f, and fix some difference "Δf" $= \delta > 0$. The Darth Vader rule (this is what normal people call the layer cake representation of the Lebesgue integral, but I was raised by probability theorists) shows that

$$\int_\Omega |f|\,\mathrm{d}x = \int_{t\geq 0} |\{f \geq t\}|\,\mathrm{d}t,$$

where $\{f \geq t\}$ is the subset $\{x \in \Omega : f(x) \geq t\}$. To approximate this integral, partition Ω into the approximate level sets

$$\Omega_i = \{x \in \Omega : i\delta \leq f(x) \leq (i+1)\delta\}$$

for $i \geq 0$. Then taking $\Delta t = \delta$, we get

$$\int_\Omega |f|\,\mathrm{d}x \approx \sum_{i\geq 0} \delta(|\{f \geq i\delta\}| - |\{f \geq (i+1)\delta\}|) = \delta \sum_{i\geq 0} |\Omega_i|.$$

For the gradient, observe that for small δ, the change in f from Ω_i to Ω_{i+1} is roughly δ times the perimeter of Ω_i, as the volume of $\Omega_i \setminus \Omega_{i+1}$ is about the perimeter scaled by this difference. This gives

$$\int_\Omega |Df|\,\mathrm{d}x \approx \sum_{i\geq 0} |\partial\Omega_i|\delta.$$

By the isoperimetric inequality we know $4\pi|\partial\Omega_i|^2 \geq |\Omega_i|$ (for simplicity, assume Ω is in $\mathbf{R}^2$). Doing some rearranging, we get

$$\sqrt{|\Omega_i|}\frac{|\partial\Omega_i|}{2\sqrt{\pi}} \geq |\Omega_i|.$$

Moreover, as Ω_i has area at most πd^2 (by the assumption that Ω had diameter d), this gives

$$|\Omega_i| \leq |\partial\Omega_i|\frac{d}{2},$$

and summing up, we get the claimed Poincaré inequality

$$\int_\Omega |f|\,\mathrm{d}x \leq \frac{d}{2}\int_\Omega |Df|\,\mathrm{d}x.$$

Of course, making this rigorous and general across dimensions is significantly harder, but as it turns out, isoperimetric inequalities and Poincaré constants are deeply connected. In fact, much recent progress in the field of convex geometry and the Kannan-Lovász-Simonovits conjecture involves establishing related Poincaré inequalities with respect to arbitrary log-concave probability measures, but that is a story for a very different time.

4. Conclusion

Overall, we've seen how Poincaré inequalities, a crucial and non-formal result in the theory of Sobolev spaces and harmonic analysis, are used critically in establishing the theory of Riemann surfaces. Moreover, the constants in such inequalities have natural connections to the geometry of their domains, and this connection is bidirectional: just as boundedness or convexity assumptions (say) on a domain can quantify the specific Poincaré constant, knowledge of Poincaré inequalities can investigate or even characterize the spectral geometry of manifolds. While an arguably classical result, many current streams of research revolve around this circle of ideas, from extending these inequalities to arbitrary measures to comparision geometry. Hopefully the reader has learned something interesting about the nature of these inequalities and their broader role within modern mathematics.

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