May's Nilpotence Thrm

— : content
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— : clarifications / references / warnings

Goal is to prove the following conjecture of May:

Thrm (Mathew - Naumann - Noel, 2015) ← conjectured in Bruner - May - McClure - Steinberger, 1986

Let R be H_∞ - ring spectrum, $x \in \pi_* R$ w/ $x \in \ker(\pi_* R \longrightarrow H\mathbb{Z}_* R)$. Then x is nilpotent.

Point: don't need complex cobordism of R, but do need theory of power operations.

Building off of Eunice & Cameron's talk, our goal is to reduce this to a statement about the behavior of $K(n)$-local H_∞-E_n-algebras.

Given time, will discuss some consequences of this.

Outline:

(1) What's up w/ $K(n)$-local H_∞-E_n-algebras?

(2) Power Operations & "λ-like" ring structures.

(3) Proof of May Nilpotence

(4) Applications

§1

Recall: $\mathrm{Alg}_{\mathbb{E}_\infty}(Sp)$ is (equiv to) the cat of algebras over the free $\mathbb{E}_\infty$-monad:

cat = ∞-cat unless homotopy cat.

$$P(X) = \bigoplus_{k \geq 0} \underbrace{\left(E_\infty(k)_+ \otimes X^{\otimes k} \right)_{h\Sigma_k}} \quad (\text{w/ } \mathcal{O} \text{ an } \mathbb{E}_\infty\text{-operad})$$

"k^{th} extended power": $D_k(X) = \left(E_\infty(k)_+ \otimes X^{\otimes k}\right)_{h\Sigma_k}$ ← maybe need to be careful abt simplicial tensoring w/ ∞-operad $\mathbb{E}_\infty$.

$E_\infty(k) \simeq pt$ w/ free $\Sigma_k \curvearrowright E_\infty(k)$ – can do internal to Sp.

This descends to a monad on hSp:

Def. An <u>H_∞-ring spectrum</u> is an algebra for $\mathbb{P}$ on hSp.
In particular, if R is H_∞, then we have maps
$D_k(R) \to R \quad \forall k$, but only coherent up to htpy.

Rmk. • Any E_∞ ring forgets down to an H_∞ struct, but converse is false! [Noel, 2010]

→ fundamentally, E_∞ is a much richer condition, but native to Sp.

- Historically, H_∞-alg were contrasted w/ commutative rings in hSp by the existence of these extended power maps;

"H_∞ = com rings in hSp + power ops"

The power op. struct is what made this a useful notion.

- Tagline: "E_∞"– homotopy coherent, in Sp

"H_∞" – homotopy commutative + power ops, in hSp

Lawson – being a homotopy $\mathcal{O}$-alg is much stronger than being

$\mathcal{O}$-alg in htpy cat:

$$[\mathcal{O}(n) \times_{\Sigma_n} X^n, X] \quad \text{v.} \quad \pi_0 \mathcal{O}(n) \to [X^n, X]$$

— Questions —

For the next 2 Sections, fix a prime p & height n.

Let $K(n)$ be (2-periodic) Morava K-theory, &

E_n = Morava E-theory

Recall: E_n admits essentially unique E_∞-struct. [Goerss-Hopkins]

Notation: $\check{E}$-alg are $K(n)$-local E_n-algebras.

Rezk calls this completed E-hom; $\check{E}(X) = L_{K(n)}(E_n \wedge X)$

Smash prod is $X \otimes_{\check{E}} Y = L_{K(n)}(X \otimes_E Y)$ on Mod_E

(Could use $L_{T(n)}$ here – telescope conjecture is true for E_n-modules – [Hovey - Strickland])

Not com. E-alg;

Have a monad on $P(X) = \bigoplus_{n \geq 0} (X^{\otimes_E n})_{h\Sigma_n}$

still equiv. tho

c.f. Rezk, Congruence, 2.7: at least in ∞-cats, $Com^{\otimes}$ is literally $E_\infty^{\otimes}$.

→ HA 5.1.1.5

Def. E_∞-algebra in Mod_{E_n} are alg. over $\mathbb{P}_E(X) = \bigoplus_{k \geq 0} \cancel{(E_n(k)_+ \otimes X^{\otimes_E k})_{h\Sigma_k}}$ $(X^{\otimes_E k})_{h\Sigma_k}$

This respects $K(n)$-equiv, so get a monad on $Mod_{\check{E}}$. $\mathbb{P}_{\check{E}}$

cst if anyone objects to notation for H_∞-$\check{E}$-alg.

→ $hAlg_{\check{E}} := hAlg_{\mathbb{P}_{\check{E}}}(Mod_{\check{E}})$.

Rmk. If $R \in hAlg$, the equiv. $\mathbb{P}_{\check{E}}(\check{E}(-)) \simeq \check{E}\,\mathbb{P}(-)$ (from equiv. nat. equiv. $\check{E}((-)^{\otimes n}) \simeq \check{E}(-)^{\otimes_{\check{E}} n}$)

makes $\check{E}(R) \in hAlg_{\check{E}}$.

n.b. $\check{E}$ does not commute w/ coproducts.

§2

Now want to discuss power operation (additive residue for multiplication operation) on H_∞-$\check{E}$-algebra

for $R \in hAlg_{\check{E}}$, $\pi_* R$ is $\mathbb{Z}/2$-graded;

$\pi_* E_n \cong Wk[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$ (u_i deg 0, u deg 2)

k perfect.

Constr.

$T \in hAlg_{\check{E}}$, $\alpha \in \check{E}_0 B\Sigma_{k+} = \pi_0(L_{K(n)}(E_n \wedge B\Sigma_{k+}))$

$Q_\alpha : \pi_0 T \longrightarrow \pi_0 T$ defined on $x : S^0 \to X$, or $E_n \to T$ in E_n-module

Should α be map of $\check{E}$-modules?

☆ $Q_\alpha(x) = S^0 \xrightarrow{\alpha} \check{E}(B\Sigma_{k+}) \simeq (E_\infty(k)_+ \otimes E_n^{\otimes_{\check{E}} k})_{h\Sigma_k} \xrightarrow{D_k(x)} (E_\infty(k)_+ \otimes T^{\otimes_{\check{E}} k}) \xrightarrow{\mu_k} T$

w/ $D_k(x)$ - functorial extended power

μ_k ~ H_∞-struct on T.

Ex. for α : Hurewicz$(\bullet \hookrightarrow B\Sigma_k) = S^0 \hookrightarrow \check{E}(B\Sigma_{k+})$,

Q_α is k^{th} power map.

consider $k=p$

Because $\check{E}_0 B\Sigma_{p+}$ is f.g. free even E_0-module, we have an iso

$$\check{E}_0 B\Sigma_{p+} \cong \mathrm{Mod}_{\pi_0 E}(E^0(B\Sigma_{p+}), \pi_0 E)$$

Q: $h\mathrm{Mod}_E$ v. $\mathrm{Mod}_{\pi_0 E}$? latter has less structure;

$h\mathrm{Mod}_{H\mathbb{F}_2}$ v. $\mathrm{Mod}_{\mathbb{F}_2}$, latter has action by a Koszul alg. of "Dyer-Lashof" ops. [May?]

Recall from Eunice's talk that the addition operation

on subgroup $\Gamma \subseteq \check{E}_0(B\Sigma_{p+})$

$$0 \to \Gamma \to \check{E}(B\Sigma_{p+}) \xrightarrow[\text{transfer maps}]{} \prod_{0<i<p} \check{E}_0((B\Sigma_i \times B\Sigma_{p-i})_+)$$

Svs. Q: Γ v. Γ^n_m?

Cohom, let $J = \bigoplus_{0<i<p} \mathrm{Im}(E^0(B\Sigma_i \times B\Sigma_{p-i})_+ \longrightarrow E^0 B\Sigma_{p+}) \trianglelefteq E^0 B\Sigma_{p+}$

Then

$$\{\text{additive operations } \Gamma\} \longleftrightarrow \left\{ \begin{array}{c} \pi_0 E\text{-lin maps } E^0(B\Sigma_{p+}) \to \pi_0 E \\ \text{factoring through quotient by } J \end{array} \right\}$$

– Question –

The magic behind May Nilpotence is following lemma, the start of which is due to Rezk.

Lemma Let $T \in h\mathrm{Alg}_{\check{E}}$. Then $\exists$ ops Q, θ on $\pi_0 T$ s.t.

(1) $(-)^p = Q(-) + p\,\theta(-)$

(2) Q is additive

(3) $\theta(0) = 0$

Rmk. This might look like a ∂-ring struct, but above height 1, Q is ~~not~~ a ring hom (at $n=1$, get Adams ops - c.f. Cameron's talk)

Pf. From [Rezk, 10.5], have following diagram of $\pi_0 E$-modules

$$\begin{tikzcd} E^0 B\Sigma_{p+} \arrow[r, "r"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\circlearrowleft"] & E^0 B\Sigma_{p+}/J \arrow[d, "\phi_2"] \\ \pi_0 E \arrow[r, "\phi_1"'] & \pi_0 E/p \end{tikzcd}$$

r, ϕ_1 quotient, ϕ_2 induced making diag commute.

ε corresponds to bpt inclusion, i.e. $Q_\varepsilon = (-)^p$.

By result of Strickland, $E^0 B\Sigma_{p+}/J$ is f.g. free,

so get a section s of r

Get additive op. $Q \leadsto \varepsilon \circ s \circ r$

$$\begin{tikzcd} E^0 B\Sigma_{p+} \arrow[r, "r"] \arrow[d, "\varepsilon"'] & E^0 B\Sigma_{p+}/J \arrow[d, "\phi_2"] \arrow[l, dashed, bend left, "s"] \\ \pi_0 E \arrow[r, "\phi_1"'] & \pi_0 E/p \end{tikzcd}$$

Now, we see that

$$\begin{aligned} \phi_1(\varepsilon - \varepsilon s r) &= \phi_1 \varepsilon (1 - sr) \\ &= \phi_2 r (1 - sr) \\ &= 0, \end{aligned}$$

Use that $\pi_0 E$ is p-torsionfree? $\to$ Will works on p-t.f.

So $\exists\ f : E^0 B\Sigma_{p+} \longrightarrow \pi_0 E$ s.t. $\varepsilon - \varepsilon s r = p \cdot f$.

Let θ be op assoc. to f; then $\leftarrow$ obstruction to $Q = (-)^p - p\theta\ (\pm)$ being ring map.

this tells us $(-)^p = Q(-) + p\theta(-)$, &

Q is additive! $(1), (2)$ ✓

Finally, if $x = 0 \in \check{E}_0 T$, $D_p(x)$ factors through

$$\begin{tikzcd} \left(E_\infty(p)_+ \otimes E_n^{\otimes_{\check{E}} p}\right)_{h\Sigma_p} \arrow[rr, "D_p(x)"] \arrow[dr] & & \left(E_\infty(p)_+ \otimes T^{\otimes_{\check{E}} p}\right)_{h\Sigma_p} \\ & \left(E_\infty(p)_+ \otimes pt^{\otimes_{\check{E}} p}\right)_{h\Sigma_p} \simeq pt \arrow[ur] & \end{tikzcd}$$

so $Q_\alpha(x) = 0\ \forall \alpha$. //

– Question –

Rmk. By a result of Ando–Hopkins–Strickland, the behavior of Γ at Morava E-theory over a height n formal group G/k is controlled by the deformation theory of G/k.

[Rezk, Congruence Criterion] & [Koszul]

In particular, Γ-modules are equiv to QCoh on a stack of G/k deformations & isogenies lifting relative Frob.

So, the $\mathcal{J}$-ring-like nature of H_∞-$\check{E}$-alg. has a "geometric" interpretation – Nat's Nov. talk.

However, as we'll see shortly, θ's usefulness is in how it behaves "like" $\mathcal{J}$ by decreasing p-adic valuation.

§3

Recall May's nilp thm:

It suffices to prove following (b/c then Hurewicz all factor through $\mathbb{Z}$-Hurewicz)

Thrm Let R be H_∞ ring, $x \in \pi_* R$, w/ $h(x)$ nilp. in $H\mathbb{Q}(R)$ & $H\mathbb{F}_p(R)$ $\forall$ all prime p.

Then x is nilp.

note: all we need for DHS nilp is that Hurewicz img is itself nilp!

why? if $h_*(x)^n = 0$, then apply to

$x^n!$

By a form of DHS nilp. [Nilp II, Thm 3.(i)],

$\{K(n)\}$ detects (p-local) nilpotence.

So have following reductions: ("enough to show")

- x nilp

$\Leftarrow$ Hurewicz img in $H\mathbb{Q}$, $H\mathbb{F}_p$, $K(n)$ nilp.

$\Leftarrow$ If x nilp in $H\mathbb{Q}$, then some pow of it is torsion.

$\Leftarrow$ By assumption, x nilp in $H\mathbb{F}_p$.

$\Leftarrow$ To show nilp in $K(n)$, suffices to show in $\check{E}_\bullet R$,

b/c we have a map $E_n \to K(n)$ $\forall$ (p,n).

– Question –

So conclusion follows from following applied to $T = \check{E}(R)$, x = Hurewicz img of $x \in \pi_\bullet R$.

Thm (2.1) || Let $T \in hAlg_{\check{E}}$, $x \in \pi_j T$.

(1) if j even, $p^m x = 0$, then $x^{(p+1)^m} = 0$.

(2) if j odd, then $x^2 = 0$

Pf. (2) is proved by Rezk [Cong., 3.14] Q: how exactly?

For (1):

a) Assume $j = 0$; $\pi_\bullet T$ is $\mathbb{Z}/2$-graded b/c E_n is even.

($u \in \pi_2 E$ either a unit in $\pi_2 T$ or $\pi_\bullet T = 0$; latter is trivial)

b) If p a nonzerodivisor in $\pi_0 T$, then ← i.e. have cancellation $px = py \Rightarrow x = y$; mult-by-$p$ map is inj. surj $\Leftrightarrow$ p-divisible, iso $\Leftrightarrow$ p-invertible. but not really lol – ring inverse need to be L&R inv by def.

$$\theta(p^m x) = p^{pm-1} x^{p-1} - p^{m-1} Q(x) \qquad \text{(b/c additive)}$$

$$= p^{m-1}\left(\left(p^{(p-1)m} - 1\right) x^p + x^p - Q(x)\right)$$ ↙ this boi is morally a unit (??)

$$= x^p \left(p^{pm-1} - p^{m-1}\right) + p^m \theta(x) \qquad (*)$$

c) If $\pi_0 T$ has p-torsion, factor $x : S^0 \to T$ as

$$\begin{tikzcd} S^0 \arrow[r, "x"] \arrow[d, "i"'] & T \\ \mathbb{P}_{\check{E}} S^0 \arrow[ur, dashed, "P(x)"'] & \end{tikzcd}$$

free H_∞-$\check{E}$-alg

$\mathbb{P}_{\check{E}} S^0$ is torsion-free, so $(*)$ holds w/ i instead of x;

Applying π_0, $P(x)$ gives a ring map sending eq. $(*)$ in $\check{E}_\bullet S^0$ to $\pi_0 T$; so $(*)$ holds in $\pi_0 T$.

d) By $(*)$ & $\theta(0) = 0$, we know

$$p^{m-1} x^p = p^{pm-1} x^p + p^m \theta(x),$$

mult by x: ↙ clearly 0; $pm - 1 > m$ if $p \geq 2$

$$p^{m-1} x^{p+1} = p^{pm-1} x^{p+1} + \cancel{p^m x}\,\theta(x) = 0$$

//

Rmk. Since $2x^2 = 0$ for x in odd deg, we do have $x^6 = 0$,

Shout graded comm alg:

Rezk, '09 Congruence Criterion, 3.14

Idea:

[3.13] Power op alg $L_{K(n)} \mathbb{P}_p(\Sigma^c E)$ is in odd degs if c is odd.

$\Rightarrow$ [3.14] || $A \in hAlg_{\check{E}}$, $x \in \pi_q A$, odd. $\Rightarrow x^2 = 0$

Pf. Only hard if (2)-local. Let $f : \Sigma^q E \to A$ rep. x. Then $x^2 \in \pi_{2q} A$ rep'd by img of $\pi_{2q} L \mathbb{P}_2(\Sigma^q E)$ under the map

in hMod$_E$ ↑ maybe Mod$_E$ → $L_{K(n)} \mathbb{P}_2(\Sigma^q E) \longrightarrow L_{K(n)} \mathbb{P}_2(\Sigma^q A) \longrightarrow A$ ↙ under the map.

but $\pi_{2q} L_{K(n)} \mathbb{P}_2(\Sigma^q E) = 0$, b/c q odd. //

but Reeh's bound is better.

$E \to$

§4

First, some low-hanging fruit:

Cor (4.1) | Let $x \in \pi_{2*+1} R$ for R an E_∞-ring, w/ nilp $H\mathbb{F}_2$ img. Then x is nilp.

Pf. In $H\mathbb{Q}$ & $H\mathbb{F}_p$ for odd primes, x^2 is 2-torsion, so 0. //

Cor (4.2) | If R is E_∞-ring w/ $0 = m \cdot 1 \in \pi_0 R$ for $m \neq 0$, then R is $K(n)$-acyclic $\forall$ (p,n).

Pf. Apply to $\check{E}_n(R)$ $\Rightarrow$ img of 1 is nilpotent in $K(n)_0 R$ //

Rmk. This is how May nilpotence pops up in §9 of Chromatic Nullstellensatz: severely constrains support of E_∞-ring.

Def. (9.1) | The chromatic support of p-local ring R is

$$supp(R) = \{ n \in \mathbb{N} \mid T(n) \otimes R \neq 0 \}$$

DNS: p-local ring R:

(1) $T(n)$-acyc $\Leftrightarrow$ $K(n)$-acyc.

(2) $R \simeq 0$ iff $T(n)_* R = 0$ $\forall n$ & $\mathbb{F}_{p*} R = 0$

By result of Burklund, for E_m, $0 \le m < \infty$, any $J \subseteq \mathbb{N}$ is realizable, MU/v_i^{m+1} is E_m-MU-alg, so

$$supp\left(\bigotimes_{MU}^{i \in \mathbb{N} \setminus J} MU/v_i^{m+1} \right) = J!$$

But May nilp implies:

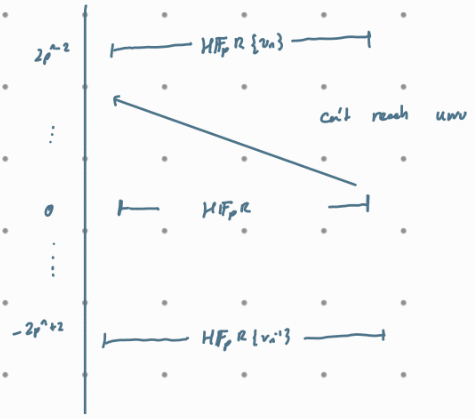
Thm | If $R \in CAlg(Sp)$ & $R \otimes \mathbb{Q} = 0$, then R is $T(n)$-acyclic $\forall n \geq 0$. If $supp(R) \neq \emptyset$, then $0 \in supp(R)$.

Hahn's then refines this: $T(n)$-acyc $\Rightarrow$ $T(n+1)$-acyc, so $supp(R)$ either $\emptyset$ or $\{0, \dots, n\} \subseteq \mathbb{N}$. n is "height of R"

Pf This obs due to Lawson; type 0 or rationally contractible.

If $H\mathbb{Q}_* R = 0$, then $\pi_* R$ all torsion, so above implies R $K(n)$-acyclic $\forall$ (n,p).

Additionally, collapse of AHSS for $\overline{K(n)}_* R$ to $(2p^n-2)$-periodic $H\mathbb{F}_{p*}(R)[v_n^{\pm 1}]$, for $n \gg 0$, forces $\mathbb{F}_p \otimes R = 0$, so $R \simeq 0$! //



$k(n)$ splits as mods of Σ's of $\overline{k(n)}$

— Stop here if fewer than 10 min left —

Other thing is use of Hopkins-Mahowald to get more tempered nilp. bound;

HM: Free (p-local?) E_2 ring R w/ $p=0$ is $H\mathbb{F}_p$.
$p \neq 2$

Thm R is E_2-ring, $x \in \pi_* R$, $px = 0$ (simple p-torsion), & nilp Hurewicz img $\pi_* R \to H\mathbb{F}_{p*} R$, then x nilp.

Pf. $R[x^{-1}]$ has $px = 0$, so $p = 0$. Then, $R[x^{-1}]$ is $H\mathbb{F}_p$-alg, & splits as generalized EM spectrum.

(uses that localization of E_n is E_n – appendix A)

Bc $H\mathbb{F}_{p*}(R[x^{-1}]) = 0$, $R[x^{-1}] \simeq 0$. //

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R[x^{-1}] \\
H\mathbb{F} \arrow[r] \arrow[urr, bend right=60] & H\mathbb{F} \otimes R[x^{-1}] = 0 \arrow[ur] &
\end{tikzcd}$$

$$\frac{1}{\sum_{i=1}^m c_i} \sum_{i=1}^m c_i^2$$

$X =$ # of [illegible] of
avg [illegible]

$\mathbb{E}X = \frac{1}{m} \sum_{i=1}^m c_i$

$\mathrm{Var}\, X > 0$

$\Rightarrow \quad \mathbb{E}X^2 > (\mathbb{E}X)^2$

$$\frac{1}{m} \sum_{i=1}^m c_i^2 > \frac{1}{m^2} \left(\sum_{i=1}^m c_i\right)^2$$

$$\Rightarrow \frac{1}{\sum_{i=1}^m c_i} \sum_{i=1}^m c_i^2 > \frac{1}{m} \sum_{i=1}^m c_i$$

↑ avg # of [illegible] per [illegible].

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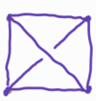

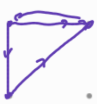

$k(n)$. $\alpha^t = 0$

$(\alpha^t)^{(m)} = 0$