THE MOTIVIC PURITY THEOREM

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Contents

The purity theorem in motivic homotopy theory provides an analog of the tubular neighborhood theorem in topology, and is central to establishing Gysin sequences and other structural results. The goal is to briefly review the unstable motivic category of spaces and prove the purity theorem. Along the way, constructions in algebraic geometry and intersection theory will be introduced to formulate the key insights of the proof.

This is an expository final paper for Math 277Z, with Elden Elmanto. All mistakes are my own; please reach out to me if you spot anything!

1. Motivic Spaces and Thom Spaces

We start by setting up some foundations of unstable motivic homotopy theory. Good references for this section are [AE16] and [Sha14].

The Nisnevich Topology. We want our motivic spaces to be certain sheaves of spaces on smooth schemes over a qcqs base S. Denote by $\mathbf{Sm}_S$ the category of finitely presented smooth schemes over S.[1] The following definition is due to Lurie:

Definition 1.1. The *Nisnevich topology* on $\mathbf{Sm}_S$ is the Grothendieck topology generated by finite covers $\{p_i \colon U_i \to X\}_{i \in I}$ where p_i are all étale and there is a finite sequence

$$\varnothing \subseteq Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_1 \subseteq Z_0 = X$$

of f.p. closed subschemes such that

$$\coprod_{i \in I} p_i^{-1}(Z_m - Z_{m+1}) \to Z_m - Z_{m+1}$$

admits a section for $0 \le m \le n-1$.

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[1]Some sources instead require that S be noetherian of finite Krull dimension, in which case this category is equivalent to smooth schemes of finite type.

When S is noetherian of finite Krull dimension, this condition is equivalent to the usual definition of a Nisnevich cover: every point x has a preimage $y \in U_i$ such that $k(x) \to k(y)$ is an isomorphism. In addition, the Zariski topology is coarser than the Nisnevich topology, which is coarser than the étale topology.

For practical purposes, one may think of the Nisnevich topology as being "generated" by *distinguished Nisnevich squares*, of the form:

$$\begin{array}{ccccc} U \times_X V & \longrightarrow & V & \longleftarrow & p^{-1}(X - U) \\ \big\downarrow & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \cong} \\ U & \overset{\circ}{\underset{i}{\hookrightarrow}} & X & \longleftarrow & X - U \end{array}$$

where p is étale, i is an open immersion, and p is an isomorphism away from U (the left square here is the actual "distinguished square"). In this situation, $\{i\colon U \to X, p\colon V \to X\}$ is a Nisnevich cover of X, as open immersions are étale and we can consider the sequence $\varnothing \subseteq X - U \subseteq X$, and $p^{-1}U \sqcup U \to U$ has a section.

The usefulness of these squares is highlighted by the following proposition:

Proposition 1.2. *Suppose S is noetherian of finite Krull dimension. A presheaf of spaces $F\colon \mathcal{S}_S^{\mathrm{op}} \to \mathcal{S}$ is a Nisnevich sheaf (i.e. satisfies descent with respect to the Nisnevich topology) iff F carries distinguished squares to Cartesian squares of spaces and $F(\varnothing)$ is final.*

Proof. See [WW19, 2.1]. □

$\mathbb{A}^1$-Local Motivic Spaces. In addition to requiring our "spaces" to be sheaves with respect to the Nisnevich topology, we also would like to invert $\mathbb{A}^1$, which is analogous forming the homotopy category of topological spaces by inverting the unit interval. Write $\mathrm{Psh}(\mathbf{Sm}_S) = \mathrm{Fun}(\mathbf{Sm}_S, \mathbf{Spc})$ for the ∞-category of presheaves of spaces on $\mathbf{Sm}_S$.

Definition 1.3. Let J be the collection of morphisms containing Nisnevich covers and projections $\mathbb{A}^1 \times_S X \to X$. The ∞-categorical localization

$$\mathbf{Spc}_S^{\mathbb{A}^1} := J^{-1}\mathrm{Psh}(\mathbf{Sm}_S)$$

is the *∞-category of motivic spaces* [Sha14].

Many sources construct the unstable motivic category as a model category via $\mathbb{A}^1$-localizing the model category of simplicial Nisnevich sheaves on $\mathbf{Sm}_S$. We therefore trust that the above formalism is understandable, and the distinction unimportant.

Example 1.4. By inverting $\mathbb{A}^1$, we find that for $p\colon E \to X$ a vector bundle in $\mathbf{Sm}_S$, p is an $\mathbb{A}^1$-equivalence (where E and X are thought of as spaces via the Yoneda embedding). In addition, sections of vector bundles are $\mathbb{A}^1$-equivalences.

While checking arbitrary maps are $\mathbb{A}^1$-equivalences is hard, it turns out there are easy ways of detecting cocartesian diagrams of spaces, thanks to Proposition 1.2.

Proposition 1.5. *If S is noetherian of finite Krull dimension, then distinguished Nisnevich squares are cocartesian squares in* $\mathbf{Spc}_S$ [AE16, 4.13].

We can define the *quotient* X/Z of a map $Z \to X$ in $\mathbf{Sm}_S$ as

$$X/Z = \mathrm{cof}(Z \to X) \in \mathbf{Spc}_S^{\mathbb{A}^1}.$$

In addition, by Proposition 1.5, if we are given a distinguished square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

we have a natural $\mathbb{A}^1$-equivalence

$$\frac{V}{U \times_X V} \to \frac{X}{U}$$

of quotients. This fact will be useful in proving the purity theorem.

2. Blow-Ups and Normal Cones

In this section, we will discuss blow-ups and normal cones, two constructions from algebraic geometry that play a crucial role in the purity theorem. Intuitively, we want an algebro-geometric analog of taking tubular neighborhoods.

Blow-Ups. Consider a closed immersion $Z \hookrightarrow X$, and let $\mathcal{I}$ be the corresponding ideal sheaf. In good cases, Z is an *effective Cartier divisor*, meaning that $\mathcal{I}$ is locally generated by a single nonzerodivisor of $\mathcal{O}_X$; equivalently, $\mathcal{I}$ is an invertible sheaf. In general, the blow-up of Z in X is an enlargement of X that corrects Z to be an effective Cartier divisor.

Definition 2.1. The *blow-up of Z in X* is a map $\pi \colon \mathrm{Bl}_Z X \to X$ fitting into a diagram

$$\begin{array}{ccccc} E_Z X & \hookrightarrow & \mathrm{Bl}_Z X & \hookleftarrow & \pi^{-1}(X - Z) \\ \downarrow & & \downarrow \pi & & \downarrow \cong \\ Z & \hookrightarrow & X & \hookleftarrow & X - Z \end{array}$$

where the left square is Cartesian, the right vertical map is an isomorphism, and the closed immersion of the *exceptional locus* $E_Z X \hookrightarrow \mathrm{Bl}_Z X$ is an effective Cartier divisor.

To construct this, one can take the graded sheaf of ideals $\bigoplus_{n \geq 0} \mathcal{I}^n$ and defines

$$\mathrm{Bl}_Z X = \underline{\mathrm{Proj}}_X \bigoplus_{n \geq 0} \mathcal{I}^n,$$

with the canonical map $\pi \colon \mathrm{Bl}_Z X \to X$. Let us check the two properties promised by our definition. We start by recording a technical lemma:

Lemma 2.2. *Let $f\colon X \to X'$ be a flat morphism of schemes, $Z' \nrightarrow X'$ a closed subscheme, and $Z = f^{-1}Z'$. Then there is a Cartesian diagram:*

$$\begin{array}{ccc} \operatorname{Bl}_Z X & \longrightarrow & \operatorname{Bl}_{Z'} X' \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow[f]{} & X' \end{array}$$

Proof. See [Sta22, 0805]. ◻

Proposition 2.3. *In the situation above:*

(a) $\pi^{-1}(X - Z) \to X - Z$ is an isomorphism.

(b) $E_Z X \nrightarrow \operatorname{Bl}_Z X$ is an effective Cartier divisor,

Proof. (a) Recall that $X - Z \hookrightarrow X$ is flat (it's an open immersion), so by the above lemma, $\pi^{-1}(X-Z) \cong \operatorname{Bl}_\varnothing X - Z$. Thus, it suffices to prove that blowing up the empty set (i.e. $\mathcal{I} = \mathcal{O}_X$) does nothing; this follows because, $\operatorname{Proj} A[T] \to \operatorname{Spec} A$ is an isomorphism [Sta22, 01MI].

(b) It suffices to show this affine-locally on X. Assume $Z = \operatorname{Spec} A/I \nrightarrow X = \operatorname{Spec} A$. By [Sta22, 0804], $\operatorname{Bl}_Z X$ is covered by $\operatorname{Spec} A[I/a]$, where $A[I/a] = (\bigoplus_{n\geq 0} I^n)_{a^{(1)}}$ is the affine blowup algebra. In this situation, we see that $IA[I/a] = aA[I/a]$ and a is a nonzerodivisor in $A[I/a]$. As $E_Z X$ in $\operatorname{Spec} A[I/a]$ corresponds to $\operatorname{Spec} IA[I/a] = \operatorname{Spec} aA[I/a]$, we see the ideal sheaf defining $E_Z X$ is invertible, as desired. ◻

The above proof also shows that the ideal sheaf of $E_Z X \nrightarrow \operatorname{Bl}_Z X$ is $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\operatorname{Bl}_Z X} = \bigoplus_{n\geq 0} \mathcal{I}^{n+1}$, which means we can identify $E_Z X$ as

$$E_Z X = \underline{\operatorname{Proj}}_X \left(\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1} \right).$$

In addition, recall that the *normal bundle* of Z in X is $N_Z X = (\mathcal{I}/\mathcal{I}^2)^\vee$, and that for a smooth embedding, the map $\operatorname{Sym}^n(\mathcal{I}/\mathcal{I}^2) \to \mathcal{I}^n/\mathcal{I}^{n+1}$ is an isomorphism. Thus, when $Z \nrightarrow X$ is smooth, there is an isomorphism $E_Z X \cong \mathbb{P} N_Z X$, where $\mathbb{P}(\mathcal{E}) = \underline{\operatorname{Proj}}_X(\operatorname{Sym} \mathcal{E}^\vee)$ is the projectivization of $\mathcal{E}$.

One source of intuition for blow-ups is that they resolve singular or otherwise "bad" loci. If we were working with manifolds, we might mimic this process by cutting out a tubular neighborhood of Z and taking a quotient of the resulting sphere bundle in $N_Z X$, thus replacing Z by a projectivization of $N_Z X$ and leaving the complement untouched. We conclude this discussion with an important example:

Example 2.4. Consider the blow-up of 0 inside $\mathbb{A}^n_k = \operatorname{Spec} A$, where $A = k[x_1, \ldots, x_n]$. Then $I = (x_1, \ldots, x_n)$, and $\operatorname{Bl}_0 \mathbb{A}^n_k = \operatorname{Proj} \bigoplus_{n\geq 0} I^n$. The surjection of graded rings

$$A[y_1, \ldots, y_n] \to \bigoplus_{n\geq 0} I^n$$

sending $y_i \mapsto x_i$ in degree 1 exhibits $\mathrm{Bl}_0 \mathbb{A}^n_k$ as a closed subscheme of $\operatorname{Proj} A[y_1, \ldots, y_n] = \mathbb{A}^n_k \times \mathbb{P}^{n-1}$, cut out by a set of homogenous polynomials defining the kernel of the above map. The polynomials $x_i y_j - x_j y_i$, for $1 \le i, j \le n$, suffice [Har77, 7.12.1].

In particular, $\mathrm{Bl}_0 \mathbb{A}^n$ is isomorphic to the total space of $\mathcal{O}(-1)$ on $\mathbb{P}^{n-1}$, and the exceptional locus is a copy of $\mathbb{P}^{n-1}$ sitting as the zero section. For $n = 2$, this results in the famous picture of a twisted "ribbon" over a plane. We'll return later to the case of blowing up the zero section of a trivial vector bundle, for which the picture is quite similar.

Deformation to the Normal Bundle. As before, the normal cone of $Z \hookrightarrow X$ is

$$C_Z X = \underline{\mathrm{Spec}}_Z \bigoplus_{n \ge 0} \mathcal{I}^n / \mathcal{I}^{n+1} \to Z$$

, and the normal bundle (as a vector bundle on Z) is

$$N_Z X = \underline{\mathrm{Spec}}_Z \left(\operatorname{Sym} \mathcal{I}/\mathcal{I}^2 \right) \to Z.$$

A key construction in intersection theory is deformation to the normal cone, which allows one to replace $Z \hookrightarrow X$ with the zero section of Z in $C_Z X$ [Ful84, 5]. This tool allows the nice properties of the normal cone to be levereged in constructing intersection products and Gysin morphisms. In our case, we need a similar construction for smooth (or even regular) embeddings which deforms $Z \hookrightarrow X$ to the zero section of Z in $N_Z X$.

Starting with $i_1 : Z \hookrightarrow X$ in $\mathbf{Sm}_S$, for a qcqs base scheme S, we want to get a family

$$\begin{array}{ccc} Z \times_S \mathbb{A}^1 & \overset{i}{\hookrightarrow} & D_Z X \\ & {\scriptstyle \pi_2} \searrow \quad \swarrow & \\ & \mathbb{A}^1 & \end{array}$$

such that the fiber at $t = 0$ is $i_0 : Z \hookrightarrow N_Z X$ and the fiber at $t = 1$ is $i_1 : Z \hookrightarrow X$.

To start, consider

$$\mathrm{Bl}_{Z \times_S \{0\}} (X \times_S \mathbb{A}^1) \xrightarrow{\pi} X \times_S \mathbb{A}^1 \to \mathbb{A}^1.$$

Over $t = 0$, the fiber is $\pi^{-1} (X \times_S \{0\})$. This contains the exceptional divisor

$$\mathbb{P} \left(N_{Z \times \{0\}} X \times \mathbb{A}^1 \right) = \mathbb{P}(N_Z X \oplus \mathcal{O})$$

along with $\mathrm{Bl}_{Z \times \{0\}} (X \times \{0\})$. These two subschemes are joined along a copy of $\mathbb{P}(N_Z X)$, which is the exceptional divisor of the latter. The former is the union of $N_Z X$, as the zero section, along with $\mathbb{P}(N_Z X)$, which is the section at ∞. We want the inclusion

$$Z \times \{0\} \to X \times \{0\} \to \mathrm{Bl}_{Z \times \{0\}} (X \times \mathbb{A}^1)$$

to be the zero section of $N_Z X$, i.e. the zero section of the exceptional divisor. So, define

$$D_Z X = \mathrm{Bl}_{Z \times_S \{0\}} (X \times_S \mathbb{A}^1) - \mathrm{Bl}_{Z \times_S \{0\}} \left(X \times_S \{0\} \right).$$

Thus, the fiber of $D_Z X \to \mathbb{A}^1$ at 0 is $\mathbb{P}(N_Z X \oplus \mathcal{O}) - \mathbb{P}(N_Z X) \cong N_Z X$, and the fiber of the closed inclusion $Z \times_S \mathbb{A}^1 \to D_Z X$ at 0 is the zero-section embedding into the normal bundle. However, at $t = 1$, $\pi^{-1}(X \times \{1\}) \cong X \times \{1\}$, so the fiber is $Z \hookrightarrow X$ as needed.

3. Proof of the Purity Theorem

Finally, we are ready to state and prove the purity theorem. Our presentation roughly follows [AE16, 7] or [Nar14]. Throughout, fix a qcqs base scheme S.

Theorem 3.1 (Purity). *Let $Z \hookrightarrow X$ be a closed embedding in $\mathbf{Sm}_S$, and let $\nu_Z \colon N_Z X \to Z$ be the normal bundle. There is an natural $\mathbb{A}^1$-equivalence*

$$\frac{X}{X - Z} \simeq \mathrm{Th}(\nu_Z).$$

Here, Th is the *Thom space*: given a vector bundle $\mathcal{E} \to X$, we define

$$\mathrm{Th}(\mathcal{E}) := \frac{\mathcal{E}}{\mathcal{E} - X},$$

where $X \hookrightarrow \mathcal{E}$ is the zero section and this is the cofiber in $\mathbf{Spc}_S^{\mathbb{A}^1}$.

Per the discussion on deformation to the normal bundle, we can consider our family

$$Z \times_S \mathbb{A}^1 \to D_Z X$$

over $\mathbb{A}^1$. Looking at the fibers over $t = 0, 1$ and taking quotients gives the morphisms

$$i_0 \colon \mathrm{Th}(\nu_Z) = \frac{N_Z X}{N_Z X - Z} \to \frac{D_Z X}{D_Z X - Z \times_S \mathbb{A}^1}$$

and

$$i_1 \colon \frac{X}{X - Z} \to \frac{D_Z X}{D_Z X - Z \times_S \mathbb{A}^1}.$$

We want to show that these two maps are always $\mathbb{A}^1$-equivalences, giving the natural equivalence in the statement of Theorem 3.1.

First, we can directly check that all is good for the case of trivial vector bundles:

Lemma 3.2. *Purity holds for the zero section $Z \hookrightarrow \mathbb{A}^n_Z = Z \times_S \mathbb{A}^n$,*

Proof. In this case, the normal bundle $N_Z \mathbb{A}^n_Z$ is the trivial bundle on Z. Also, the blowup of $\mathbb{A}^n_Z \times \mathbb{A}^1$ at the zero section $Z \times \{0\}$ is isomorphic to the total space of $\mathcal{O}(-1)$ over $\mathbb{P}^n_Z$, and the image of $\mathrm{Bl}_Z \mathbb{A}^n_Z$ is the same bundle over a copy of $\mathbb{P}^{n-1}_Z$ embedded at ∞. Thus,

$$D_Z \mathbb{A}^n Z \cong \mathcal{O}(-1)\big|_{\mathbb{A}^n_Z},$$

and the map $D_Z \mathbb{A}^n_Z \to \mathbb{A}^n$ is an $\mathbb{A}^1$-bundle. The deformation family $Z \times_S \mathbb{A}^1 \to D_Z \mathbb{A}^n_Z$ is then the inclusion of the fiber over the zero section of our original trivial bundle. Clearly, as the fibers over 0 and 1 are both fibers of a vector bundle, we find that i_0 and i_1 are $\mathbb{A}^1$-equivalences, as desired. $\square$

As it turns out, all closed immersions reduce to this case.

Definition 3.3. A *smooth pair* (X, Z) is a closed embedding $Z \nrightarrow X$ in $\mathbf{Sm}_S$. A map of smooth pairs $(X, Z) \to (X', Z')$ is a Cartesian square

$$\begin{array}{ccc} Z & \hookrightarrow & V' \\ \downarrow & & \downarrow \\ Z' & \hookrightarrow & X'. \end{array}$$

Such a map is *Nisnevich* if $f : X \to X'$ is étale and $f^{-1}Z' \to Z'$ is an isomorphism.

Note that a map of smooth pairs being *Nisnevich* implies the left square

$$\begin{array}{ccccc} U \times_{X'} X & \longrightarrow & X & \longleftarrow & f^{-1}Z' \\ \downarrow & & \downarrow f & & \downarrow \simeq \\ U := X' - Z' & \hookrightarrow & X' & \hookleftarrow & Z' \end{array}$$

is a distinguished square.

Smooth pairs can be locally characterized by the following, due to Grothendieck.

Proposition 3.4. *Let $i : Z \nrightarrow X$ be a smooth pair, with codimension c along Z. Then there is a Zariski cover*

$$\{U_i \to X\}_{i \in I}.$$

and set of étale morphisms

$$\left\{U_i \to \mathbb{A}^{n_i}_S\right\}_{i \in I}$$

such that for all i, there is linear inclusion $\mathbb{A}^{n_i - c}_S \to \mathbb{A}^{n_i}_S$ giving rise to a Cartesian square

$$\begin{array}{ccc} U_i \times_X Z & \longrightarrow & U_i \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{A}^{n_i - c}_S & \longrightarrow & \mathbb{A}^{n_i}_S. \end{array}$$

Proof. See [GR02, II.4.10]. □

In addition, call a map of smooth pairs $(X, Z) \to (X', Z')$ *weakly excisive* if the square

$$\begin{array}{ccc} Z & \longrightarrow & \dfrac{X}{X - Z} \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & \dfrac{X'}{X' - Z'} \end{array}$$

is cocartesian in $\mathbf{Spc}^{\mathbb{A}^1}_S$. As $Z \to \mathbb{A}^1_Z$ has contractible cofiber, showing the maps

$$(N_Z X, Z) \xrightarrow{i_0} (D_Z X, \mathbb{A}^1_Z) \xleftarrow{i_1} (X, Z)$$

of smooth pairs are weakly excisive suffices to prove the purity theorem. We start with the following lemmas: Say that *purity holds* for (X, Z) if i_0, i_1 as above are weakly excisive.

Lemma 3.5. *Say (X,Z) is a smooth pair and $\{U_i \to X\}$ is a Zariski cover such that purity holds for*

$$(U_{i_1} \times_X \cdots \times_X U_{i_n}, Z \times_X U_{i_1} \times_X \cdots \times_X U_{i_n})$$

for all tuples $i_1, \dots, i_n$. Then purity holds for (X,Z).

Proof. Write $(U_\bullet, Z_\bullet) \to (X,Z)$ for the simplicial object associated to this cover. By assumption on i_1 we have a simplicial cocartesian diagram

$$\begin{array}{ccc} Z_\bullet & \longrightarrow & U_\bullet/(U_\bullet - Z_\bullet) \\ \downarrow & & \downarrow \\ Z_\bullet \times_S \mathbb{A}^1 & \longrightarrow & D_{Z_\bullet}U_\bullet/(D_{Z_\bullet}U_\bullet - Z_\bullet \times_S \mathbb{A}^1) \end{array}$$

Taking geometric realization gives the cocartesian square on the right,

$$\begin{array}{ccc} |Z_\bullet| & \longrightarrow & |U_\bullet/(U_\bullet - Z_\bullet)| \\ \downarrow & & \downarrow \\ |Z_\bullet \times_S \mathbb{A}^1| & \to & |D_{Z_\bullet}U_\bullet/(D_{Z_\bullet}U_\bullet - Z_\bullet \times_S \mathbb{A}^1)| \end{array} \quad \simeq \quad \begin{array}{ccc} Z & \longrightarrow & X/(X-Z) \\ \downarrow & & \downarrow \\ Z \times_S \mathbb{A}^1 & \to & D_Z X/(D_Z X - Z \times_S \mathbb{A}^1) \end{array}$$

which shows i_1 is weakly excisive for (X,Z). The proof is similar for i_0. □

For the next, note the following observations of [Hoy17, 3.19],

$$(X,Z) \xrightarrow{f} (X',Z') \xrightarrow{g} (X'',Z'')$$

are morphisms of smooth pairs, then

(a) when f is weakly excisive, g is weakly excisive iff gf is, by [Lur09, 4.4.2.1], and

(b) if $g\colon Z' \to Z''$ is an $\mathbb{A}^1$-equivalence and g and gf are weakly excisive, then f is weakly excisive, as

$$\frac{Z'}{Z} \simeq \frac{Z''}{Z} \simeq \frac{X''/(X''-Z'')}{X/(X-Z)} \simeq \frac{X'/(X'-Z')}{X/(X-Z)},$$

where the last equivalence holds because in the respective square for Z, contractibility of $\mathrm{cof}(Z' \to Z'')$ implies $X''/(X''-Z'') \simeq X'/(X'-Z')$.

Lemma 3.6. *If $(V,Z) \to (X,Z)$ is a Nisnevich morphism of smooth pairs, then purity holds for (V,Z) iff it holds for (X,Z).*

Proof. First, observe that Nisnevich morphisms of smooth pairs are weakly excisive. Recall that if $U \coloneqq X - Z$, then

$$\begin{array}{ccc} U \times_X V & \hookrightarrow & V \\ \downarrow & \lrcorner & \downarrow \\ U & \longrightarrow & X \end{array}$$

is a distinguished square, so by Proposition 1.5, is cocartesian. Therefore, we have a tautological equivalence $Z = Z$ and an equivalence

$$\frac{V}{V - Z} = \frac{V}{U \times_X V} \xrightarrow{\simeq} \frac{X}{U} = \frac{X}{X - Z}.$$

The cofibers of both these maps are contractible, as needed.

Now, consider the diagram

$$\begin{array}{ccccc} (N_Z V, Z) & \xrightarrow{i_0} & (D_Z V, \mathbb{A}^1_Z) & \xleftarrow{i_1} & (V, Z) \\ \downarrow & & \downarrow & & \downarrow \\ (N_Z X, Z) & \xrightarrow[i_0]{} & (D_Z X, \mathbb{A}^1_Z) & \xleftarrow[i_1]{} & (X, Z) \end{array}$$

As all the vertical arrows are Nisnevich morphisms, they are weakly excisive. The lemma follows from observation (a) above: if for example, the maps on the top row are weakly excisive, then $(N_Z V, Z) \to (N_Z X, Z)$ and the composite

$$(N_Z V, Z) \to (D_Z V, \mathbb{A}^1_Z) \to (D_Z X, \mathbb{A}^1 X)$$

are weakly excisive, so the bottom i_0 is excisive. The remaining cases are similar. □

We now have the tools to prove that purity holds for all smooth pairs (X, Z).

Proof. By Lemma 3.5 and considering a cover $\{U_i \to X\}$ as in Proposition 3.4, it suffices to show that if $(X, Z) \to (\mathbb{A}^n, \mathbb{A}^m)$ is a map of smooth pairs with $X \to \mathbb{A}^n$ étale, then purity holds for (X, Z). Note that this map is *not* in general Nisnevich.

Following [MV99, 2.28], let $c = n - m$ and define $Z \times_S \mathbb{A}^c \to \mathbb{A}^n$ to be the product of $Z \to \mathbb{A}^m$ with $\mathrm{id}_{\mathbb{A}^c}$. Because the former map is étale, we know

$$Z \times_{\mathbb{A}^m} Z \subseteq X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c),$$

the fiber of the projection $X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c) \to \mathbb{A}^n$ over $\mathbb{A}^m$, is the disjoint union of the diagonal embedding $Z \to Z \times_{\mathbb{A}^m} Z$ and a closed subscheme $Y \nrightarrow X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c)$. Define $U \coloneqq X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c) - Y$; now projection gives étale maps $p\colon U \to X$ and $q\colon U \to Z \times \mathbb{A}^c$ such that $p^{-1}Z \cong Z$ and $q^{-1}Z \cong Z$.[2] Thus, we have Nisnevich maps $(U, Z) \to (X, Z)$ and $(U, Z) \to (Z \times_S \mathbb{A}^c, Z)$. Purity holds for $(Z \times_S \mathbb{A}^c, Z)$ by Lemma 3.2, so by Proposition 3.6, purity holds for (U, Z), whence it holds for (X, Z), thus completing the proof. □

[2]Note that the fiber over Z of the projection $X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c) \to X$ is $Z \times_{\mathbb{A}^m} Z$ because $(X, Z) \to (\mathbb{A}^n, \mathbb{A}^m)$ is a map of smooth pairs, and likewise for $q^{-1}Z$; in effect we are "correcting" this fiber so that we can replace this map with a Nisnevich map of smooth pairs.

Intuition and Applications. One source of intuition for this theorem is to consider Z and X being smooth manifolds. Then, the tubular neighborhood theorem states that $N_Z X$ embeds as an open subset of X:

$$\begin{array}{ccccc} N_Z X - Z & \hookrightarrow & N_Z X & \longrightarrow & \dfrac{N_Z X}{N_Z X - Z} \\ \downarrow & & \downarrow & & \downarrow \cong \\ X - Z & \hookrightarrow & X & \longrightarrow & \dfrac{X}{X - Z} \end{array}$$

Thus, this is the motivic analog of the tubular neighborhood theorem.

This is used in defining, for instance, the Gysin sequence. In the setting of intersection theory, one can define *flat pullback* on algebraic cycles associated to a flat map $f\colon X \to Y$ of relative dimension r:

$$\begin{aligned} f^*\colon A_k Y &\longrightarrow A_{k+r} X \\ [Z] &\longmapsto [f^{-1} Z]. \end{aligned}$$

When f is the projection map of a vector bundle $\pi\colon E \to X$ (and r is the rank of E over X),then flat pullback is an isomorphism $\pi^*\colon A_{k-r} X \to A_k E$ [Ful84, 3.3]. Thus, we can define a Gysin map $A_k E \to A_{k-r} X$ associated to the zero section $s\colon X \to E$ by $s^*(\beta) = (\pi^*)^{-1}\beta$.

Deformation to the normal bundle allows for a generalization in the case of a regular embedding $i\colon Z \to X$ of codimension r: the Gysin homomorphism is then the composite

$$A_k X \to A_k N_Z X \xrightarrow{s^*} A_{k-r} Z,$$

where the first map is constructed using this deformation, and the second is the Gysin homomorphism associated to a vector bundle.

In topology, the tubular neighborhood theorem along with the Thom isomorphism gives an isomorphism $\overline{H}^{i-r}(Z;k) \to \overline{H}^{i}(\mathrm{Th}(\nu_Z);k)$, which can be used to construct the Gysin sequence. Thus, as one expects, a major consequence of the purity theorem is in establishing the Gysin long exact sequence in motivic homotopy theory.

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