FAST TIMES AT $\mathcal{M}_{fg}$

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Contents

The theory of formal groups both has historical connections to various domains of algebraic geometry and number theory, as well as a fundamental centrality to the chromatic picture of stable homotopy theory. The goal is to introduce formal groups, discuss how one thinks about and works with these objects, define $\mathcal{M}_{fg}$, the moduli stack of formal groups, and explain key results on the global structure of this stack. While we will approach this in fair generality, at times applications towards chromatic homotopy theory will be highlighted.

This is an expository final paper for a tutorial on stacks and moduli, with Taeuk Nam. All mistakes are my own; reach out to me if you spot anything!

1. An Introduction to Formal Schemes

First Impressions. Recall that a "ring" is a commutative, unital ring. Write $\mathbf{CAlg}_R$ for the category of commutative R-algebras. To us, an *R-scheme* is a functor $\mathbf{CAlg}_R \to \mathbf{Set}$ satisfying Zariski descent and with an open cover by affine schemes.[1]

In any case, our discussion of formal groups necessitates that we first step into the world of formal schemes. This provides the right framework to capture fine infinitesimal details.

Definition 1.1. A *formal scheme* is a small filtered colimit of affine schemes.

As a brief categorical remark, the category of formal schemes has all small colimits and the inclusion of schemes into formal schemes preserves finite colimits [Str00, Prop. 4.7].

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[1] Some sources, notably [Pst20], require the stronger condition of satisfying étale descent, but the difference will be minor for our usage; most important phenomona will be defined Zariski-locally anyway.

Now for some motivating examples. Let $Z = \operatorname{Spec}(A/I) \hookrightarrow X = \operatorname{Spec}(A)$ be a closed embedding of affine R-schemes. The *formal completion* of X at Z is

$$X_Z^\wedge := \operatorname{colim}_n \operatorname{Hom}(A/I^n, -).$$

That is, the B-points of $X_Z^\wedge$ are B-points of X which kill some power of I; this constitutes infinitesimal fuzz around Z. If we equip A with the I-linear topology, then this colimit is equivalently *continuous* homomorphisms from A to a R-algebra B, where the latter is given the discrete topology. Thus, we can fortuitiously think of formal schemes as schemes which "care about the topology," motivating the following.

Definition 1.2. Let A be an R-algebra equipped with an I-linear topology, for some ideal $I \leq A$. The *formal spectrum* $\operatorname{Spf}(A)$ is the sheaf of affine R-schemes given by

$$\operatorname{Spf}(A)(B) := \operatorname{Hom}^{\text{cts}}(A, B) \cong \operatorname{colim}_n \operatorname{Hom}(A/I^n, B).$$

Note that the formal spectrum only depends on the topology on A, not on the particular ideal; for instance, we could always replace I by I^2, or replace A by it's completion $\widehat{A} = A_I^\wedge$ with the limit topology, and not change anything. With this, we define *formal affine n-space* over $\operatorname{Spec} R$ as $\widehat{\mathbb{A}}_R^n := \operatorname{Spf}(R[[x_1, \ldots, x_n]])$, where $R[[x_1, \ldots, x_n]]$ is given the $\mathfrak{m} = (x_1, \ldots, x_n)$-adic topology.

To better grasp this definition, consider the difference between $\mathbb{A}_k^1$, $\operatorname{Spec} k[[x]]$, and $\widehat{\mathbb{A}}_k^1 := \operatorname{Spf} k[[x]]$ for k a field. $\mathbb{A}_k^1$ is of course a line over $\operatorname{Spec} k$, and we should think of $\operatorname{Spec} k[[x]]$ as a formal disk around the origin of $\mathbb{A}_k^1$. However, for a k-algebra B, the B-points of $\widehat{\mathbb{A}}_k^1$ are

$$\widehat{\mathbb{A}}_k^1(B) = \operatorname{Hom}^{\text{cts}}(k[[x]], B) = \{b \in B \mid b \text{ nilpotent}\}.$$

Thus, while $\operatorname{Spec} k[[x]]$ is like a formal disk (in terms of Zariski spectra, this has one generic point corresponding to the disk around the origin), whereas $\widehat{\mathbb{A}}_k^1$ is just the infinitesimal fuzz around the origin. One should think of this as being *much* smaller.

Miscellaneous Useful Results. Let us note some facts which will be useful later. Recall that for a Zariski sheaf $(X, \mathcal{O}_X)$, the *ring of global sections* is $\Gamma(X, \mathcal{O}_X) = \operatorname{Hom}_{sh}(X, \mathbb{A}^1)$. If $X = \operatorname{Spec} R$ is affine, then $\Gamma(X, \mathcal{O}_X) \cong R$, and this characterizes affine schemes.

Proposition 1.3. *Let A be a topological ring with the I-adic topology. Then we have*

$$\Gamma(\operatorname{Spf}(A), \mathcal{O}_{\operatorname{Spf}(A)}) \cong A_I^\wedge.$$

Proof. Recall the presentation of $\operatorname{Spf}(A) \cong \operatorname{colim}_n \operatorname{Spec}(A/I^n)$. Thus, we have

$$\operatorname{Hom}_{sh}(\operatorname{Spf} A, \mathbb{A}^1) \cong \lim_n \operatorname{Hom}_{sh}(\operatorname{Spec} A/I^n, \mathbb{A}^1) \cong \lim_n A/I^n = A_I^\wedge.$$

□

In fact, this gives a canonical monomorphism $\operatorname{Spf} A \to \operatorname{Spec} \widehat{A}$; we recognize this for $A = k[[x]]$ as the inclusion of the infinitesimal neighborhood of 0 into the formal disk.

Proposition 1.4. *Let A be a I-complete R-algebra. There is a bijection*

$$\operatorname{Hom}_{\operatorname{Spec} R}(\operatorname{Spf} A, \widehat{\mathbb{A}}^1_R) \cong \{x \in A | x^n \to 0\}.$$

(The right-hand side are "topologically nilpotent.")

Proof. By the previous proposition, maps $\operatorname{Spf} A \to \mathbb{A}^1_R$ are in bijection with A, because A is complete; thus it suffices to see which of these factor the monomorphism $\widehat{\mathbb{A}}^1_R \to \mathbb{A}^1_R$. By our presentation of $\operatorname{Spf} A$, we see that $x \in A$ determines a map to $\widehat{\mathbb{A}}^1_R$ if it is nilpotent modulo I^n for all n; a quick sanity check shows this is equivalent to the condition $x^n \to 0$ in the adic topology. □

Corollary 1.5. *Maps from $\widehat{\mathbb{A}}^n_R := \left(\widehat{\mathbb{A}}^1_R\right)^{\times n} \to \widehat{\mathbb{A}}^1_R$ are in bijection with elements of the ideal of power series in $R[[x_1, \dots, x_n]]$ with nilpotent constant term.*

Proof. We can describe $\widehat{\mathbb{A}}^n_R$ as $\operatorname{Spf} R[[x_1, \dots, x_n]]$, where the latter ring is viewed in the $\mathfrak{m} = (x_1, \dots, x_n)$-adic toplogy. The topologically nilpotent elements in this ring are precisely those with nilpotent constant term. □

2. Formal Groups, Formal Group Laws, and Lie Algebras

As one might expect, a formal group is but a group object in formal schemes. For our purposes, we'll be slightly more restrictive: we want our formal groups to be commutative and resemble the tangent spaces of 1-dimensional Lie groups.[2]

Definition 2.1. A *formal group* over $\operatorname{Spec} R$ is a commutative group object in formal schemes which is Zariski-locally isomorphic to $\widehat{\mathbb{A}}^1_R$ as a formal scheme.

The goal for the remainder of this paper is to develop tools for how to think of these objects and explain some of what is known about their moduli and applications.

Formal Group Laws. Fix a ring R, viewed as a base $\operatorname{Spec} R$. Let us first consider the case of formal groups G over R which are isomorphic as formal schemes to $G \cong \widehat{\mathbb{A}}^1_R = \operatorname{Spf} R[[x]]$. Note that the group structure, given by maps

$$\operatorname{Spf} R[[x]] \times_{\operatorname{Spec} R} \operatorname{Spf} R[[x]] \to \operatorname{Spf} R[[x]]$$
$$\operatorname{Spec} R \to \operatorname{Spf} R[[x]]$$

come from R-algebra maps

$$R[[x]] \to R[[x, y]]$$
$$R[[x]] \to R.$$

[2]Apparently, Tomer Schlank has at one point tried to consider the applicability of 2-dimensional formal groups to homotopy theory, but for trivial reasons this isn't that much more effective.

The first map is determined by the image of x, which is a power series $F \in R[[x,y]]$, and the second sends $x \mapsto 0$. By convention, we write $a +_F b = F(a,b) \in B$ for the group addition on the B-points of G (which are just nilpotent elements of B). Note that the group axioms (actually, just the axioms of a commutative monoid) now require that

(1) $a +_F b = b +_F a$,

(2) $a +_F 0 = a$,

(3) $a +_F (b +_F +c) = (a +_F b) +_F c$.

This implies F has the form $x + y$ + higher order terms. In addition, a power series is invertible under composition if and only if it has zero constant term and unit linear term and either this or the fact that the B-points of G are nilpotents implies such an addition on $G(B)$ has inverses; i.e. it defines an honest sheaf of abelian groups.

Definition 2.2. Any power series $F \in R[[x,y]]$ satisfying conditions (1)-(3) above is called a *formal group law* over R. The *formal group associated to* F is $G_F := \operatorname{Spf} R[[x]]$, with the group structure on $G_F(B)$ given by $+_F$.

The preceding discussion and Corollary 1.5 show that all formal group structures on $\widehat{\mathbb{A}}^1_R$ arise as G_F for some f.g.l. F. Thus, all formal group structures on $\widehat{\mathbb{A}}^1_R$ are *coordinatizable.*

Some examples of formal group laws are the *additive formal group law*, $F_a(x,y) = x + y$ and the *multiplicative formal group law* $F_m(x,y) = (1+x)(1+y) = x + y + xy$. These both have arise naturally in chromatic homotopy theory.

In addition, we define for every $n \in \mathbf{Z}$ the *n-series* of F, $[n]_F(x) \in R[[x]]$, which corresponds by Corollary 1.5 to the map $\times n\colon G_F \to G_F$. These will meow a reappearance.

Lie Algebras and Coordinatizability. With this in hand, we can give a more concrete equivalent definition of formal groups.

Definition 2.3. A *formal group* over $\operatorname{Spec} R$ is a Zariski sheaf $G\colon \mathbf{CAlg}_R \to \mathbf{Ab}$ such that there exist $r_1 + \cdots + r_n = 1 \in R$ with $G \cong G_{F_i}$ over $\operatorname{Spec} R[r_i^{-1}]$.

Thus, formal groups are Zariski-locally coordinatizable, but not all formal groups are coordinatizable globally. For example, let $\mathcal{L} \to \operatorname{Spec} R$ be a nontrivial line bundle, and let $\widehat{\mathcal{L}}$ be the formal completion along the zero section, with the addition inherited from $\mathcal{L}$. This is locally isomorphic to $G_a := G_{F_a}$, but is only globally isomorphic to such if $\mathcal{L}$ was trivial. In general, the obstruction to coordinatizability is having nontrivial "transition maps," and the *Lie algebra* of a formal group allows us to measure this obstruction.

Definition 2.4. The *Lie algebra* of a formal group G over $\operatorname{Spec} R$ is

$$\mathfrak{g} = \ker\left(G(R[\epsilon]/\epsilon^2) \to G(R)\right).$$

Notice that for any $\lambda \in R$, the map $\epsilon \mapsto \lambda\epsilon$ determines an R-module structure on $\mathfrak{g}$. In addition, for $G = G_F$, $\mathfrak{g} \cong \epsilon R[\epsilon]/\epsilon^2 \cong R$ as R-modules. Because this is true Zariski-locally for any G, $\mathfrak{g}$ is always an invertible R-module, giving rise to a line bundle on $\operatorname{Spec} R$.

Proposition 2.5. *This line bundle is trivial if and only if G is coordinatizable.*

Proof. See [Lur10, 11.7]. The key observation is that the obstruction to G being coordinatizable is that the Zariski-local isomorphisms $G \cong G_{F_i}$ may not be compatible; given a trivialization $\mathfrak{g} \cong R$ however, we can choose them to be compatible. □

Remark 2.6. Another construction associated to a formal group G over R which is useful to consider is the *sheaf of invariant differentials*, which is a quasicoherent sheaf on $\operatorname{Spec} R$. By [Pst20, 5.16], this is isomorphic to the pullback of the relative cotangent complex $\Omega^1_{G/\operatorname{Spec} R}$ along the 0 section, so in a sense this is the dual Lie algebra. The power of invariant differentials, besides detecting when a formal group is coordinatizable, is in providing an explicit isomorphism of formal group laws in characteristic zero to F_a. We will return to this result later, but will black-box the technology.

When first presented in isolation (or with the chromatically-minded motivation of defining a good theory of the first Chern class for a complex-orientable cohomology theory), formal group laws can seem ill-motivated. However, this presentation as arising from a choice of coordinate on $\widehat{\mathbb{A}}^1_R$ makes their nature more apparent. One should imagine the passage from formal groups to coordinatizable formal groups as that from schemes to affine schemes, and from coordinatizable formal groups to f.g.l.s as choosing generators.

3. Isomorphisms, Representability, and $\mathcal{M}_{fg}$

Our invariant definition of formal groups means we get a good notion of homomorphisms and isomorphisms of formal groups for free.Here, we want to discuss the corresponding notion for formal group laws, as well as the result that the moduli *space* of formal groups is affine.

Isomorphisms of Formal Groups and FGLs.

Definition 3.1. A *homomorphism* of formal group laws $\varphi\colon F \to G$ over R is a power series $\varphi(x) \in R[[x]]$ such that

$$\varphi(a +_F b) = \varphi(a) +_G \varphi(b).$$

φ is an *isomorphism* if φ is an invertible (under composition) power series, in which case

$$G(x, y) = \varphi(F(\varphi^{-1}(x), \varphi^{-1}(y))).$$

It is clear that if φ invertible, thus inducing an automorphism of $\operatorname{Spf} R[[x]]$ as a formal scheme, it gives rise to an isomorphism of formal groups $G_F \to G_G$ over $\operatorname{Spec} R$.

Lazard's Theorem and $\mathcal{M}_{fg}$. Define the presheaf of sets FGL as follows:

$$\mathrm{FGL}(R) = \{F \in R[[x, y]] \mid F \text{ is a formal group law}\}.$$

Given a map $f\colon R \to S$ of rings, we can pushforward a formal group law F over R to a f.g.l. f_*F over S by applying f to the coefficients. This gives an isomorphism

$$f^*G_F \coloneqq G_F \times_{\operatorname{Spec} R} \operatorname{Spec} S \cong G_{f_*F},$$

along with the action of FGL on morphisms.

Theorem 3.2. *FGL is an affine scheme.*

Proof. Consider the ring $\mathbf{Z}[a_{ij}]$ with generators for every pair of nonnegative integers (i,j), and let $F(x,y) = \sum a_{ij}x^i y^j$. Let I be the ideal generated by the coefficients of $F(F(x,y),z) - F(x,F(y,z))$, $F(x,y) - F(y,x)$, and $F(x,0) - x$; then F defines a formal group law on $L \coloneqq \mathbf{Z}[a_{ij}]/I$. For any ring R and power series $F_R \in R[[x,y]]$ over R, there is a ring homomorphism $f\colon \mathbf{Z}[a_{ij}] \to R$ such that $f_*F = F_R$; this passes uniquely to $f\colon L \to R$ iff F_R is a formal group law. Thus, we get $\operatorname{Spec} L \cong \mathrm{FGL}$. □

The ring L (and the isomorphism $\operatorname{Spec} L \cong \mathrm{FGL}$, i.e. a universal f.g.l. F) corepresenting FGL is called the Lazard ring. This ring L has a much simpler description:

Theorem 3.3 (Lazard)**.** *There is an isomorphism* $L \cong \mathbf{Z}[x_1, x_2, \ldots]$ *with* $|x_i| = 2i$.

Proof. Omitted: see [Lur10] for a combinatorial proof, [Hop99] for a homological one, and [Pst20] for a deformation-theoretic approach. The key observation is that in characteristic 0, every formal group law is isomorphic to the additive one, by a unique strict isomorphism. A technical lemma known as the Symmetric Cocycle Lemma reduces this theorem to showing an isomorphism on the graded pieces of the module of indecomposables of L, which can be done via deformation theory. As a consequence, this isomorphism is far from explicit. □

By the discussion above, $\operatorname{Spec} L \cong \mathrm{FGL} \cong \mathbb{A}^\infty$ admits an action by the (affine) group scheme $\mathbb{G}_{inv}$ of invertible power series:

$$\mathbb{G}_{inv}(R) = \left\{ \sum_{i \geq 1} a_i x^i \in R[[x]] \mid a_1 \in R^\times \right\}.$$

This group has a semidirect decomposition as "strict" power series

$$\mathbb{G}_{inv}^s(R) = \left\{ x + \sum_{i \geq 2} a_i x^i \in R[[x]] \right\},$$

and $\mathbb{G}_m$;

$$\mathbb{G}_{inv} = \mathbb{G}_{inv}^s \rtimes \mathbb{G}_m.$$

As we will discuss shortly, strict power series are precisely isomorphisms of formal groups which "preserve the coordinate." Meanwhile, the $\mathbb{G}_m$-action represents different choices of coordinate, and corresponds to an (even) grading.[3]

Clearly, two f.g.l.s $F, G \in \mathrm{FGL}(R)$ are isomorphic iff there is some $\varphi \in \mathbb{G}_{inv}(R)$ such that $\varphi \cdot F = G$. Isomorphic f.g.l.s give rise to isomorphic formal groups, inspiring the following definition/theorem:[4]

[3] By convention, this grading is even because of the Koszul sign rule.

[4] Depending on the source.

Theorem 3.4. *The moduli stack of formal groups $\mathcal{M}_{fg}$ is the sheaf of spaces sending $R \in$ **CAlg** to the groupoid of formal groups over R and their isomorphisms.*

There is a morphism $FGL \to \mathcal{M}_{fg}$ sending $F \mapsto G_F$, which is faithfully flat, affine, and

$$FGL \times_{\mathcal{M}_{fg}} FGL \cong \operatorname{Spec} L \times \mathbb{G}_{inv}.$$

Thus, $FGL \to \mathcal{M}_{fg}$ presents $\mathcal{M}_{fg}$ as the quotient stack $[\mathbb{G}_{inv} \backslash FGL]$.

One way to interpret the first isomorphism is that an R-point of the displayed pullback is a pair of formal group laws F, G over R along with an isomorphism between them; this is the same by the previous discussion as a single formal group F over R along with the isomorphism φ sending F to G.

For a proof that the morphism FGL $\to \mathcal{M}_{fg}$ gives an atlas of $\mathcal{M}_{fg}$, see [Pst20]. Note that $\mathcal{M}_{fg}$ is not an algebraic stack in the traditional sense – $\operatorname{Spec} L$ is infinite-dimensional. However, $\mathcal{M}_{fg}$ is a fpqc stack (i.e. it has a fpqc atlas), and it can be recovered as a pro-algebraic stack. A key preliminary result is the following:

Proposition 3.5. $\mathcal{M}_{fg} \times \operatorname{Spec} \mathbf{Q} \cong B\mathbb{G}_m \times \operatorname{Spec} \mathbf{Q}$.

Proof. In characteristic 0, every formal group is locally isomorphic to G_a, and the automorphism of that group are $\mathbb{G}_m$. To see the former fact, note that for any FGL F over a $\mathbf{Q}$-algebra, the "logarithm"

$$\ell(x) = \int \frac{1}{F_y(x,y)}\, dx$$

exists, giving an isomorphism $G_F \to G_a$. □

Thus, the interesting questions arise when considering what happens at a prime p.

Quillen's Theorem and MU_*. Although this is not a review of the vast body of work connecting the geometry of $\mathcal{M}_{fg}$ to chromatic homotopy theory, a few remarks are warranted. The story starts with Quillen's theorem: that there is an isomorphism of the Lazard ring L with MU_*. Specifically, MU_* supports a universal formal group law.

Classically, there is the notion of a *complex-orientable cohomology theory E_**, which one might motivate as a multiplicative generalized cohomology theory with a choice of Thom class for complex vector bundles. In particular, these support a good theory of the first Chern class. A computation with the Atiyah-Hirzeburch spectral sequence shows that $E^*\mathbf{CP}^\infty = E^*\mathrm{BU}(1) \cong E_*[[t]]$, with $|t| = 2$; a choice of such an isomorphism is called a complex orientation, and is in fact equivalent to a (homotopy class of a) multiplicative map of spectra $MU \to E$. Thus, one can study the behavior of E_* by looking at the associated formal group $\operatorname{Spf} E^*\mathbf{CP}^\infty$. It turns out that this formal group is always coordinatizable, and the arithmetic of the associated formal group law encodes a vast amount of information that can classify such cohomology theories. In essence, the geometry of $\mathcal{M}_{fg}$ controls the behavior of the homotopy category of spectra.

4. Global Structure: The Height Filtration

As mentioned, the rational picture of $\mathcal{M}_{fg}$ is rather boring. Thus, for the next two sections, fix a prime p; our goal in this section is to understand $(\mathcal{M}_{fg})_{(p)} = \mathcal{M}_{fg} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. Thus, all rings considered are $\mathbf{Z}_{(p)}$-algebras.

Heights of Formal Group Laws. We have the following concrete definition:

Definition 4.1. Let F be a f.g.l. over R. Let v_n be the coefficient of x^{p^n} in the p-series $[p]_F(x)$. F has *height* $\geq n$ if $v_0 = p, v_1, \ldots, v_{n-1}$ are all 0, and height exactly n if v_n is invertible.

The height of a p-local f.g.l. measures how far away multiplication by p is from being an isomorphism. For example, F has height 0 if p is invertible, and height at least 1 if $p = 0$. In fact, the additive formal group law has p-series

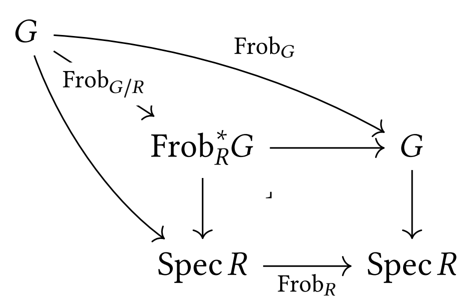
$$[p]_{F_a}(x) = px = 0,$$

and so has height ∞, while the multiplicative formal group law has p-series

$$[p]_{F_m}(x) = (1+x)^p - 1 = x^p + \text{lower order terms},$$

and so has height 1. Thus, at a prime p, the additive and multiplicative formal groups are not isomorphic.

It is perhaps worthwhile to give a more invariant description of where this notion of height comes from. Recall that for R an $\mathbf{F}_p$-algebra we have a Frobenius homomorphism $\operatorname{Frob}(r) = r^p$. Taking colimits of affines, we know that given a formal group G over R, we have an absolute Frobenius $\operatorname{Frob}_G : G \to G$ fitting into the diagram below. Taking the induced map into the pullback gives the *relative Frobenius* $\operatorname{Frob}_{G/R} : G \to G$ which is a map over $\operatorname{Spec} R$.



One might think of this as the Frobenius adjusted by "pretwisting" the coefficients. As an example, for the case of $X = \widehat{\mathbb{A}}^1_R \to \operatorname{Spec} R$, we have a very concrete description of $\operatorname{Frob}_{X/\operatorname{Spec} R}$: it is induced by the map of rings

$$\begin{gathered} R \otimes_R R[[x]] \cong R[[x]] \longrightarrow R[[x]] \\ (s \otimes \sum a_i x^i) \mapsto \sum s a_i^p x^i \longmapsto \sum s a_i^p x^{ip}. \end{gathered}$$

In particular, for G_F a formal group associated to a f.g.l. $F(x, y) = \sum a_{ij} x^i y^j$ over R, we have an isomorphism

$$\mathrm{Frob}_R^* G_F \cong G_{F'}$$

where $F'(x, y) = \sum a_{ij}^p x^i y^j$.

With all this in place, we say that a formal group G over an $\mathbf{F}_p$-algebra is of height at least n if multiplication by p factors (uniquely) through the n-th relative Frobenius. If this factorization is an isomorphism, G has height exactly n.

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Frob}^n_{G/R}} & (\mathrm{Frob}^n_R)^* G \\ & \searrow^{\times p} & \downarrow \\ & & G \end{array}$$

This description makes clear why the formal additive group has infinite height – the multiplication by p map is zero, and so factors through arbitrarily high powers of relative Frobenius.

Proposition 4.2. *Every formal group over a field k has a height $0 \leq n \leq \infty$. Over a field of characteristic p, there exists a formal group law of height n for every $1 \leq n \leq \infty$.*

Now, recall that for any formal group law F over R, the p-series is

$$[p]_F(x) = px + \text{higher degree terms},$$

and if this map factors through the n-th relative Frobenius, we should be able to write this power series as $[p]_F(x) = \varphi(x^{p^n})$ for a (unique) $\varphi(x) \in R[[x]]$. Being of height exactly n tells us that φ is invertible. Now, letting v_n be the coefficients of x^{p^n} in the p-series, we recover the original definition of height.

The Height Filtration. The vanishing of the elements v_n as defined above cuts out closed substacks of $\mathcal{M}_{fg_{(p)}}$. Importantly, even though these elements depend on the p-series of a f.g.l., the ideal they generate is an invariant of the formal group associated to the f.g.l., thus allowing for the following definitions:

Definition 4.3. Let $\mathcal{M}_{fg}^{\geq n} := \operatorname{Spec}\left(L_{(p)}/(p, v_1, \ldots, v_{n-1})\right)/\mathbb{G}_{inv} \nrightarrow \mathcal{M}_{fg}$ be the closed substack of formal groups of height $\geq n$. Let the open complement of $\mathcal{M}_{fg}^{\geq n+1}$ be $\mathcal{M}_{fg}^{\leq n} \hookrightarrow \mathcal{M}_{fg}$. Finally, let $\mathcal{M}_{fg}^{=n} = \mathcal{M}_{fg}^{\geq n} \cap \mathcal{M}_{fg}^{\leq n}$ be the substack of formal groups of height exactly n.

This gives rise to the *height* filtration, also known as the *chromatic* filtration, of $\mathcal{M}_{fg_{(p)}}$ by closed substacks.

$$\mathcal{M}_{fg_{(p)}} \longleftarrow\!\!\shortmid\!\!\hookrightarrow \mathcal{M}_{fg}{}^{\geq 1} \longleftarrow\!\!\shortmid\!\!\hookrightarrow \mathcal{M}_{fg}{}^{\geq 2} \longleftarrow\!\!\shortmid\!\!\hookrightarrow \mathcal{M}_{fg}{}^{\geq 3} \longleftarrow\!\!\shortmid\!\!\hookrightarrow \ldots \longleftarrow\!\!\shortmid\!\!\hookrightarrow \mathcal{M}_{fg}{}^{\geq \infty}$$

One key result here is the following theorem of Lazard:

Theorem 4.4. *Let G_1, G_2 be two formal groups over k of height n, with $\operatorname{char} k = p$. Then there is a separable extension $f\colon k \hookrightarrow K$ such that $f^*G_1 \cong f^*G_2$. In particular, any two formal groups of the same height are isomorphic over a separably closed field.*

Proof. See [Goe08, 5.26]. ◻

Thus, for $1 \le n \le \infty$, $\mathcal{M}_{fg}{}^{=n}$ has a single geometric point represented by the formal group Γ_n over $\operatorname{Spec} \mathbf{F}_p$ of height n, and is equivalent as a stack to

$$\mathcal{M}_{fg}{}^{=n} \simeq B\operatorname{Aut}(\Gamma_n).$$

The automorphisms of this point are controlled by the *Morava stabilizer group*.

Overall, the global picture of $\mathcal{M}_{fg}$ is rather simple; zoomed-out, there is one geometric point for every pair of a prime p and height $1 \le n \le \infty$. The local picture – analyzing $\widehat{\mathcal{M}_{fg}{}^{=n}_{(p)}}$ – is far richer, involving Lubin and Tate's work on local class field theory and providing a deep understanding of isomorphisms and deformations of formal groups at finite heights. As a payoff, one gets intrinsic characterizations of objects like Morava K-theory and E-theory, which are fundamental players in the chromatic picture of stable homotopy theory, leading to various active veins of modern research.

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