A CONDENSED INTRO TO CONDENSED MATH

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Contents

Condensed mathematics is a formalism for constructing abelian categories of "topological" objects, and shows up in the computation of [BSSW24] as a means of accessing continuous cohomology. In this talk, we will provide an overview of the information from condensed math necessary for the rest of the seminar, covering condensed sets/abelian groups, solidity and its connection to continuous cohomology, and p-complete and p-adic complexes of condensed abelian groups.

These are notes from a talk I gave for Babytop, organized by Ishan Levy, Piotr Pstragowski, and Andy Senger. All mistakes are my own; please reach out to me if you spot anything!

1. Condensation

Broadly speaking, condensed mathematics is about doing algebra on objects which carry a natural topology. For our seminar, this means coming up with a nice theory of continuous group cohomology $H^i_{\mathrm{cts}}(G, C)$ as the derived functor of something, but there's a simpler example: the identity map of topological abelian groups:

$$(\mathbf{R}, \text{discrete topology}) \to (\mathbf{R}, \text{Euclidean topology}).$$

This map is not an isomorphism, but it has zero kernel and cokernel, which is sad. By replacing topological spaces with *condensed sets*, we can dispel this sadness.

This talk closely follows [BSSW24, 3.2-3.4], with many of the proofs referenced from [CS19]. In what follows, all set-theoretically issues will be (safely) ignored.

Definition 1.1. The *pro-étale site of a point* $*_{\text{pro}-\text{ét}}$ is the category of profinite sets, where the covers are jointly surjective continuous maps.

Recall that, unlike finite limits of sets, profinite limits inherit a natural topology, that being the subspace topology of the (compact) infinite product topology. That's why requiring *continuous* maps as covers makes sense. The actual *topology* of these sets is, depending on who you ask,

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pathological. These are sometimes known as *Stone spaces*, and are equivalently compact totally disconnected Hausdorff spaces.

Definition 1.2. For $\mathscr{C}$ a category of blah, the *category of condensed blah* is the category of $\mathscr{C}$-valued sheaves on $*_{\text{pro-ét}}$: denote this by $\mathbf{Cond}(\mathscr{C})$.

Example 1.1. There is a faithful functor from $\mathbf{Spc} \to \mathbf{Cond}(\mathbf{Set})$ given by sending

$$X \mapsto \underline{X} = C_{\text{cts}}(-, X).$$

This is fully faithful when restricted to compactly generated topological spaces, and can be naturally upgraded to a functor from topological groups/rings/etc. to condensed groups/rings/etc.

Example 1.2. Let $\mathbf{R}_{\text{disc}}$ denote $\mathbf{R}$ with the discrete topology. Then the map $\underline{\mathbf{R}_{\text{disc}}} \to \underline{\mathbf{R}}$ of condensed abelian groups is injective, and has a cokernel Q, which is has underlying abelian group $Q(*) = 0$. However, for a general $S \in *_{\text{pro-ét}}$, this is

$$Q(S) = \frac{\{\text{continuous maps } S \to \mathbf{R}\}}{\{\text{locally constant maps } S \to \mathbf{R}\}} \neq 0.$$

Just in case you were losing faith in the world for whatever reason, rest assured that the category of condensed abelian groups should fill you with hope:

Proposition 1.3. $\mathbf{Cond}(\mathbf{Ab})$ *is a (co)complete closed symmetric monoidal abelian category. That is/moreover:*

(a) We have an internal hom $\underline{\mathrm{Hom}}(\mathscr{M}, \mathscr{N})$ *and the adjunction*

$$\mathrm{Hom}(\mathscr{M} \otimes \mathscr{N}, \mathcal{P}) \simeq \mathrm{Hom}(\mathscr{M}, \underline{\mathrm{Hom}}(\mathscr{N}, \mathcal{P})).$$

(b) There is a "free condensed abelian group" functor from $\mathbf{Cond}(\mathbf{Set}) \to \mathbf{Cond}(\mathbf{Ab})$ *sending* X *to* $\mathbf{Z}[X]$*, which is left adjoint to the forgetful functor.*

(c) $\mathbf{Cond}(\mathbf{Ab})$ *has enough projectives (being of the form* $\mathbf{Z}[S] = \mathbf{Z}[\underline{S}]$ *for* S *a small extremally disconnected set), we can form the derived (*∞*-)category* $D(\mathbf{Cond}(\mathbf{Ab}))$*, with its usual derived tensor product and derived internal hom* RHom.

Proof. See [CS19, 1.10] and [CS19, 2.2]. □

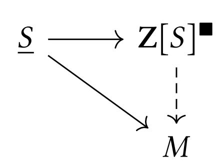
2. Solidification

When doing topology, we like to work with complete objects, like $\mathbf{R}$, $\mathbf{Z}_p$, or $\mathbf{Q}_p$, but these are usually not closed under tensor products (e.g. what's the topology on $\mathbf{Z}_p \otimes \mathbf{R}$?). In the condensed setting, this is known as being "solid."

Definition 2.1. Let $S = \varprojlim S_i$ be a profinite set, with S_i finite. The *free solid abelian group on* S is

$$\mathbf{Z}[S]^{\blacksquare} = \varprojlim \underline{\mathbf{Z}[S_i]},$$

where $\mathbf{Z}[S_i]$ is the (discrete topological) free abelian group on S_i. A *solid abelian group* is a condensed abelian group M such that for all profinite S, the following diagram has a unique extension:



Proposition 2.2. *The category* **Solid** *of condensed abelian groups is a (co)complete abelian subcategory of* $\mathbf{Cond}(\mathbf{Ab})$*. The induced functor* $D(\mathbf{Solid}) \to D(\mathbf{Cond}(\mathbf{Ab}))$ *is fully faithful, and the following are equivalent:*

(a) $C \in D(\mathbf{Cond}(\mathbf{Ab}))$ *lies in the essential image of* $D(\mathbf{Solid})$*.*

(b) $H^i(C)$ *is a solid abelian group for all* $i \in \mathbf{Z}$*.*

(c) For all profinite sets S*, the natural map is an isomorphism:*

$$\mathrm{RHom}(\mathbf{Z}[S]^{\blacksquare}, C) \to \mathrm{RHom}(\mathbf{Z}[S], C).$$

Moreover, the inclusion $\mathbf{Solid} \hookrightarrow \mathbf{Cond}(\mathbf{Ab})$ *admits a left adjoint, "solidifcation"* $M \mapsto M^{\blacksquare}$*, which is the unique colimit-preserving extension of* $\mathbf{Z}[S] \mapsto \mathbf{Z}[S]^{\blacksquare}$*, and there is a unique symmetric monoidal tensor* $\otimes^{\blacksquare}$ *for which solidifcation is symmetric monoidal.*

Proof. See [CS19, 5.8,6.2]. □

Luckily, we see that being complete isn't far off from being solid:

Lemma 2.3. *If* M *is a topological abelian group which is separated and complete for a linear topology, then* $\underline{M}$ *is solid.*

Proof. By hypothesis, there is a cofiltered system of open subgroups $M_n \subset M$ inducing this topology, with $M \to \varprojlim M/M_n$ an isomorphism. Now, if $S = \varprojlim S_i$, with S_i finite, and $f\colon S \to M$ continuous, then for each n, the map $S \to M/M_n$ is locally constant, so it factors through $S_{i(n)} \to M/M_n$. Then, extend this to a map $\underline{\mathbf{Z}[S_i]} \to \underline{M/M_n}$, and take the limit in n, furnishing the desired map $\mathbf{Z}[S]^{\blacksquare} \to \underline{M}$ □

3. Continuous Cohomology

According to a previous talk, the reason condensed mathematics shows up in this work on (ostensibly) $K(n)$-local homotopy theory is to give a good notion of continuous cohomology. Here we discuss this connection:

Definition 3.1. Let G be a condensed group, M a condensed abelian group. A G-action on M is a map $G \times M \to M$ satisfying the usual diagrams.

In this setting, we can upgrade Lemma 2.3:

Lemma 3.2. *Let* M *be a topological abelian group, separated and complete for a linear topology, with a continuous action of a profinite group* G*. Then* $\underline{M}$ *is a solid abelian group with* $\underline{G}$*-action.*

Proof. We already know $\underline{M}$ is solid, and we want to upgrade the G-module structure to an action $\underline{G} \times \underline{M} \to \underline{M}$. It thus suffices to produce a functorial action $C_{\mathrm{cts}}(-, G) \curvearrowright C_{\mathrm{cts}}(-, M/M_n)$. The idea is that because G acts continuously, there's an index N and open subgroup $H \subset G$ with $HM_N \subset M_n$, which lets us write down an explicit factorization. See [BSSW24, 3.3.1] for details. □

Let $\mathbf{Solid}_G$ denote the (abelian) category of solid abelian groups admitting an action of $\underline{G}$, for G profinite.

Theorem 3.3. *Let G be a profinite group, and consider the fixed-points functor* $\mathbf{Solid}_G \to \mathbf{Solid}$ *defined as $M \mapsto M^G$ (the right adjoint to the trivial action functor). Let $R\Gamma(G,-)$ be its derived functor $D(\mathbf{Solid}_G) \to D(\mathbf{Solid})$ and let $H^i(G,-) = R^i\Gamma(G,-)$.*

(a) For any $C \in D(\mathbf{Solid})$, $R\Gamma(G,C)$ is the totalization of the cosimplicial object

$$n \mapsto \underline{\mathrm{RHom}(\mathbf{Z}[G^n],C)}.$$

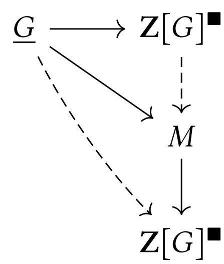
(b) If M satisfies the hypotheses of Lemma 3.2, then

$$H^i(G,\underline{M}) \simeq \underline{H^i_{\mathrm{cts}}(G,M)}.$$

Proof. See [BSSW24, 3.3.2]. The first point is that the "solid Iwasawa algebra"

$$\mathbf{Z}[G]^{\blacksquare} = \varprojlim_{H\subset G \text{ open}} \underline{\mathbf{Z}[G/H]}$$

is projective in $\mathbf{Solid}_G$. To see this, let $M \to \mathbf{Z}[G]^{\blacksquare}$ be a surjection from a solid $\underline{G}$-module, and note that to construct a section, it suffices to construct a map $\underline{G} \to M$ such that $\underline{G} \to M \to \mathbf{Z}[G]^{\blacksquare}$ is the canonical map:



To do this, let $m \in M(*)$ lift the identity section $* \to \mathbf{Z}[G]^{\blacksquare}$, and use the map $g \mapsto gm$. Likewise, the algebras $\mathbf{Z}[G^n]^{\blacksquare}$ are projective.

Thus, we have the usual simplicial resolution of the trivial $\underline{G}$-module $\underline{Z}$ in $\mathbf{Solid}_G$:

$$\ldots \substack{\to\\\to\\\to} \mathbf{Z}[G^2]^{\blacksquare} \rightrightarrows \mathbf{Z}[G]^{\blacksquare} \to \underline{\mathbf{Z}}.$$

Now (a) is easy: if $C \in D(\mathbf{Solid}_G)$, then $R\Gamma(G,C)$ is the totalization of the cosimplicial object coming from applying $\underline{\mathrm{RHom}}(-,C)$ to the above simplicial resolution. As C is solid, we know $\underline{\mathrm{RHom}}(\mathbf{Z}[G^n]^{\blacksquare},C) \simeq \underline{\mathrm{RHom}}(\mathbf{Z}[G^n],C)$, whence the claim.

For (b), notice that for $C = \underline{M}$ and $\mathbf{Z}[G]$ being free, we have

$$\underline{\mathrm{RHom}}(\mathbf{Z}[G^n],\underline{M}) \simeq \underline{\mathrm{RHom}}(G^n,\underline{M}) \simeq \underline{C_{\mathrm{cts}}(G^n,M)}.$$

Thus, $R\Gamma(G,\underline{M})$ is the totalization of the *condensed* cosimplicial resolution which normally computes continuous cohomology; ergo, $H^i(G,\underline{M})$ is "condensed" $H^i_{\mathrm{cts}}(G,M)$. □

Definition 3.4. For G profinite, let $C \mapsto C^{hG} := R\Gamma(G,C)$ be the derived fixed points functor in Theorem 3.3. For M satisfying the conditions of Lemma 3.2, write M^{hG} for $\underline{M}^{hG}$.

Theorem 3.3 shows that for "nice" M, continuous cohomology $H^*_{\text{cts}}(G, M)$ is indeed the derived functor of G-fixed points in an abelian category. Moreover, we have an honest complex M^{hG} of solid abelian groups which computes continuous cohomology. This lets us use derived categories for future computations.

Remark 3.1. Here are two examples of the previous principle:

1. If $H \subset G$ is a closed normal subgroup, $M^{hH} \in D(\mathbf{Solid}_{G/H})$, so we have a quasi-isomorphism

$$M^{hG} \simeq \left(M^{hH}\right)^{hG/H}.$$

   This formally gives the Hochschild-Serre spectral sequence

$$H^i_{\text{cts}}\left(G/H, H^j_{\text{cts}}(H, M)\right) \implies H^{i+1}_{\text{cts}}(G, M).$$

2. For $M, N \in D(\mathbf{Solid}_G)$, with G acting trivially on M, there is a *projection formula*: i.e. a canonical isomorphism in $D(\mathbf{Solid})$

$$M \otimes^{\blacksquare} N^{hG} \xrightarrow{\simeq} (M \otimes^{\blacksquare} N)^{hG}.$$

4. p-Adic Completion

To end, let's recall the notion of derived p-completeness in the context of $D(\mathbf{Ab})$.

Definition 4.1. An object $A \in D(\mathbf{Ab})$ is *(derived) p-complete* if $\mathrm{RHom}(B, A) \simeq 0$ for all $B \in D(\mathbf{Ab})$ with $B \otimes \mathbf{Z}/p \simeq 0$. $D(\mathbf{Ab})_p$ is the full subcategory of $D(\mathbf{Ab})$ of p-complete objects.

For those familiar, $D(\mathbf{Ab})_p$ is the Bousfield localization of $D(\mathbf{Ab})$ at the object $\mathbf{Z}/p$; it suffices to check that $\mathrm{RHom}(\mathbf{Z}[p^{-1}], A) = 0$. Thus, there is a left adjoint $A \mapsto \widehat{A}$ to the inclusion $D(\mathbf{Ab})_p \subset D(\mathbf{Ab})$, given by

$$\widehat{A} = \varprojlim \operatorname{cof}(A \xrightarrow{p^n} A),$$

and a unique symmetric monoidal tensor $\widehat{\otimes}$ making $A \mapsto \widehat{A}$ symmetric monoidal.

Definition 4.2. An object $A \in D(\mathbf{Cond}(\mathbf{Ab}))$ is *p-complete*, and is in the full subcategory $D(\mathbf{Cond}(\mathbf{Ab}))_p$, if for all $B \in D(\mathbf{Cond}(\mathbf{Ab}))$ with $B \otimes \mathbf{Z}/p \simeq 0$, we have $\underline{\mathrm{RHom}}(B, A) = 0$.

Lemma 4.3. *Consider $D(\mathbf{Cond}(\mathbf{Ab}))_p$: p-complete complexes of condensed abelian groups:*

(a) $D(\mathbf{Cond}(\mathbf{Ab}))_p$ is closed under limits.

(b) For $A \in D(\mathbf{Cond}(\mathbf{Ab}))_p$, if $A/p \simeq 0$, then $A \simeq 0$.

(c) The inclusion $D(\mathbf{Cond}(\mathbf{Ab}))_p \subset D(\mathbf{Cond}(\mathbf{Ab}))$ admits a left adjoint

$$A \mapsto A^{\wedge}_p = \varprojlim A/p^n.$$

Proof. See [BSSW24, 3.4.4]. □

We'd also like to know which objects in $D(\mathbf{Cond}(\mathbf{Ab}))$ "come from" $D(\mathbf{Ab})$.

Definition 4.4. Let $\Gamma_*\colon \mathbf{Cond}(\mathbf{Ab}) \to \mathbf{Ab}$ be the underlying abelian group functor $\Gamma_* A = A(*)$. This is right adjoint to $\Gamma^*\colon A \mapsto \underline{A}$. Both are exact, and are identified with their derived functors. The *discretization* of $A \in D(\mathbf{Cond}(\mathbf{Ab}))$ is

$$A^\delta := \Gamma^*\Gamma_* A = \underline{A(*)}.$$

A is *discrete* if the counit $A^\delta \to A$ is an isomorphism.

Lemma 4.5.

(a) *Discretization is t-exact, i.e. if $A \in D(\mathbf{Cond}(\mathbf{Ab}))^\heartsuit$, then $A^\delta \in D(\mathbf{Cond}(\mathbf{Ab}))^\heartsuit$.*

(b) *Discrete objects are closed under fibers, cofibers, and retracts.*

(c) *The image of $\Gamma^*\colon D(\mathbf{Ab}) \to D(\mathbf{Cond}(\mathbf{Ab}))$ consists of discrete objects.*

Proof. (a) Γ^* is given by pulling back sheaves, and Γ_* is corepresented by $\underline{Z}$, which is a projective condensed abelian group, so both are t-exact. (b) follows from this.

(c) is equivalent to Γ^* being fully faithful, which is equivalent to the unit $\mathrm{id} \to \Gamma_*\Gamma^*$ being an equivalence, which is always true. □

Definition 4.6. An object $A \in D(\mathbf{Cond}(\mathbf{Ab}))$ is *p-adic* if it is p-complete and $A \otimes \mathbf{Z}/p$ is discrete.

Proposition 4.7. *Consider the condensed p-adic completion for $A \in D(\mathbf{Ab})$: $A \mapsto (\Gamma^* A)^\wedge_p$. This factors through a (symmetric monoidal) equivalence of $D(\mathbf{Ab})_p, \widehat{\otimes}$ and the p-adic objects of $D(\mathbf{Cond}(\mathbf{Ab}))_p, \otimes^{\blacksquare}$.*

Proof. First, this functor factors through $A \mapsto \underline{A^\wedge_p}$, as

$$(\Gamma^* A^\wedge_p)^\wedge_p = \varprojlim \underline{A^\wedge/p/p^n} \simeq \varprojlim A/p^n \simeq (\Gamma^* A)^\wedge_p.$$

To see that $(\Gamma^*)^\wedge_p$ is p-adic. It is p-complete as

$$\underline{\mathrm{Hom}}(\underline{\mathbf{Z}[1/p]}, (\Gamma^* A)^\wedge_p) = \varprojlim \underline{\mathrm{Hom}}\left(\underline{\mathbf{Z}[1/p]}, \Gamma^*(A)/p^n\right) \simeq 0$$

and $(\Gamma^* A)^\wedge_p/p \simeq \Gamma^*(A/p)$ is discrete. The quasi-inverse functor is Γ_*, on p-adic objects.

Finally, $(\Gamma^* A)^\wedge_p$ is solid as a limit of (solid) discrete objects, and the same trick shows that this functor is symmetric monoidal. □

Overall, we see that if A is a derived p-complete abelian group with a continuous action of a profinite group G, then Theorem 3.3 shows that the continuous cohomology $H^i_{\mathrm{cts}}(G, A)$ is the cohomology of the continuous fixed points $\left((\Gamma^* A)^\wedge_p\right)^{hG}$, and similarly for complexes fo derived p-complete continuous G-modules. This will be useful later.

Acknowledgments

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