Complex Bordism & Quillen's Thm

(Lurie § 5-7)

1) Complex Bordism
2) Orientation
3) Homology of MU

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1) Complex Bordism:

Recall $\xi \in \mathrm{Vect}_n(X)$, WLOG, it has a metric.

twisted E. - cohom.

$$E^{\bullet - \xi}(X) := E^{\bullet}(B(\xi), S(\xi))$$

Ex. $\xi : \mathbb{R}^n \times X \longrightarrow X$

$B(\xi) = B^n \times X$, $S(\xi) = S^{n-1} \times X$,

$\Rightarrow E^{\bullet - \xi}(X) \cong E^{\bullet - n}(X)$

Note: E mult $\Rightarrow$ $E^{\bullet - \xi}(X) \in \mathrm{Mod}_{E^\bullet X}$

Def. $\xi \in \mathrm{Vect}_n(X)$, E mult. cohom thy. (ring spectrum)

E-orientation of ξ to be

$u \in E^{n-\xi}(X)$ s.t.

noncanonical.

(*) $\forall x \in X$, $x^*(u) \in E^{n-\xi_x}(\{x\}) \simeq E^0(\{x\}) = \pi_0 E$

$x^*(u)$ generates $\pi_0 E$ (as a $\pi_0 E$-module).

Say u is a Thom class for ξ in E-cohom.

Cor. $E^\bullet(X) \xrightarrow{\times u} E^{\bullet + n - \xi}(X)$.

Goal (1) All COCT have canonical E-orientation on any complex vector bundle.

(2) $\exists$ ring spectrum MU s.t. $MU \to E$ classify E-orientation.

Pf. (1) Suffices to consider $\xi : EU(n) \to BU(n)$:

$S^{2n-1} \to BU(n-1) \longrightarrow BU(n)$

$\Rightarrow S(\xi) \simeq BU(n-1)$. $D(\xi) = BU(n)$

so, $E^{\bullet - \xi}(BU(n)) \simeq E^0(BU(n), BU(n-1))$

$E^\circ(BU(n)) = E_\circ [\![c_1, \dots, c_n]\!]$

$\theta : E^\circ(BU(n)) \to E^\circ(BU(n-1))$ is surj.

$$c_i \mapsto c_i$$

$$\Rightarrow \quad E^\bullet(BU(n), BU(n-1)) = c_n \, E_\bullet[\![c_1, \ldots, c_n]\!] = (c_n).$$

<u>Claim</u> $c_n \in E^{2n}(BU(n), BU(n-1))$ is Thom class.

WTS $\forall x \in BU(n)$, $(\ast)$ holds, b/c $BU(n)$ cted,

so pick one point:

$$\begin{tikzcd} f^\bullet EU(n) \arrow[r, hook] \arrow[d] & EU(n) \arrow[d, "\xi"] \\ BU(1)^n \arrow[r, hook, "f"] & BU(n) \end{tikzcd}$$

$$c_n \mapsto E^{\bullet - f^\bullet \xi}(BU(1)^n) \cong (t_1 \cdots t_n) \subseteq \pi_0 E[\![t_1, \ldots, t_n]\!] = E^\bullet(BU(1)^n)$$

$$c_n \mapsto t_1 \cdots t_n$$

$f^\bullet \xi = \bigoplus_{i=1}^n p_i^\bullet \mathcal{O}(1)$ on $BU(1) = \mathbb{C}P^\infty$.

In base case $n=1$,

$$E^{\bullet - \xi}(BU(1)) \cong \tilde{E}^\bullet(\mathbb{C}P^\infty)$$

$u \in \tilde{E}^\bullet(\mathbb{C}P^\infty)$ is orientation of $\mathcal{O}(1)$

iff it maps to a unit in $\tilde{E}^2(S^2) = \pi_0 E$.

By constr. $u \mapsto 1$. //

In general,

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "\xi'"'] & EU(n) \arrow[d, "\xi'"] \\ X \arrow[r, "f"] & BU(n) \end{tikzcd} \qquad \xi' = f^\bullet \xi$$

define $u_{\xi'} \in E^{2n-\xi'}(X) = f^\bullet(u)$, $\left(f^\bullet : E^{2n-\xi}(BU(n)) \to E^{2n-\xi}(X)\right)$

$\Rightarrow$ Our system of Thom classes is

natural, by construction.

<u>Def.</u> $\forall n \geq 0$,

$$MU(n) := \Sigma^{-2n} BU(n)^{\xi_n}, \quad \xi_n \text{ univ bundle of rank } n.$$

$$\xi_n|_{BU(n-1)} = 1 \oplus \xi_{n-1} \Rightarrow$$

$$MU(n-1) = \Sigma^{2-2n} BU(n-1)^{\xi_{n-1}} = \Sigma^{-2n} BU(n-1)^{\overset{\xi_n}{\overset{\shortparallel}{1+\xi_{n-1}}}}$$

$$\longrightarrow \Sigma^{-2n} BU(n)^{\xi_n} = MU(n).$$

<u>Note:</u> $c_n \in E^n(BU(n)/BU(n-1)) \leftrightarrow \phi_n : MU(n) \to E.$

$$\underline{\Sigma^{-2n}} BU(n)^{\xi_n} = \underline{\Sigma_+^\infty}^{-2n} BU(n)^{\xi_n}$$

$\mathbb{S} = MU(0) \to MU(1) \to \cdots$

Def. $MU = \operatorname{colim} MU(n)$.

Aside: MU has a geometric interpretation as

$\pi_n MU \simeq$ Group of Bordism classes of n-dim'l mflds w/ stable almost cplx struc. (Thom-Pontryagin thm)

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2) Let E cplx-oriented cohom thry.

$\{\text{ring maps } MU \to E\} \leftrightarrow \{\text{complex orientations of } E\}$

Q: What is ring struct on MU?

$BU(a) \times BU(b) \to BU(a+b)$ classifies direct sum of vector bundles.

$\Rightarrow MU(a) \otimes MU(b) \xrightarrow{m_{a,b}} MU(a+b)$

$MU(n-1) \to MU(n)$ same as

$$\begin{tikzcd}
MU(n-1) = \mathbb{S} \times MU(n-1) = MU(0) \times MU(n-1) \arrow[d] \\
MU(n) \quad \quad MU(1) \times MU(n-1) \arrow[l, "m_{1,n-1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
MU(0) = \mathbb{S} = \Sigma^{\infty-2} S^2 \arrow[d] \arrow[d, shift left=8] \\
MU(1) = \Sigma^{\infty-2} \mathbb{C}P^\infty
\end{tikzcd}$$

classifying the complex orientation.

$\Rightarrow$ taking colimit, $MU \otimes MU \to MU$,

& fact MU is E_∞-ring.

Recall: E COCT, $\forall n$ c_n orientation of ξ_n

$\leftrightarrow \phi_n : MU(n) \to E$.

s.t.

1) $\phi_1 : MU(1) \to E$ is cplx orientation of E.

2) ϕ_n mult:

$$\begin{tikzcd}
MU(m) \otimes MU(n) \arrow[r] \arrow[d, "\phi_m \otimes \phi_n"'] & MU(m+n) \arrow[d, "\phi_{m+n}"] \\
E \otimes E \arrow[r] & E
\end{tikzcd}$$

3)
$$\begin{tikzcd}
MU(n) \arrow[d] \arrow[dr, "\phi_n"] & \\
MU(n+1) \arrow[r] & E
\end{tikzcd}$$

ϕ_{n+1}

$\Rightarrow$ 1) $\phi : MU \to E$ map of ring spectra

2) $\underset{\Sigma^{\infty-2}\mathbb{C}P^\infty}{MU(1)} \hookrightarrow MU \in \widetilde{MU}^2(\mathbb{C}P^\infty) \ni t,$

t is canonical orientation of MU;

$\phi(t)$ to cplx orientation of E.

$\Rightarrow \underset{\text{ring maps}}{\{MU \to E\}} \xleftrightarrow{\sim} \{\text{orientation of } E\}$

$\phi \longmapsto \phi(t)$

$\exists!\, \phi',\ \phi'(t) = t' \longleftarrow t'$

MU is universal COCT.

Really want to connect $MU \rightsquigarrow$ formal grp.

Thm (Quillen)

$L \to MU_*$ is iso & respects univ. formal grp law.

Milnor

Additionally, by same algebra.

Prop. E is COCT, $\{b_i\} \subseteq E_*(MU(1))$ dual to basis $\{t^{i+1}\}$ of $(t) \subseteq \pi_* E[[t]]$. Then img of $b_i \in E_* MU$ determine ring iso: $\pi_* E[b_1, b_2, \dots] \cong E_* MU$.

Magic scam: $E_* MU(n) \cong \mathrm{Sym}^n E_* BU(1)$.

Cor.

$H_*(MU, \mathbb{Z}) = \mathbb{Z}[b_1, b_2, \dots]$ //