ABELIAN VARIETIES AND THE SERRE-TATE THEOREM

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Contents

The theory of abelian varieties is a natural source for many p-divisible groups, giving an algebro-geometric inroad to TAF. The goal of this talk is to briskly introduce the main concepts in this theory and then discuss the Serre-Tate theorem, which states that the deformation theory of abelian varieties are controlled by the deformation theory of their associated p-divisible group. We will first define abelian varieties and isogenies, discuss the construction of the dual of an abelian variety, introduce some aspects of the theory of p-divisible groups, and end with an overview of the proof of the Serre-Tate theorem.

These are notes from a talk I gave for Babytop, organized by Ishan Levy and Andy Senger. All mistakes are my own; please reach out to me if you spot anything!

1. Abelian Varieties and Isogenies

Let k be a field; we'll largely work in $\mathbf{Sch}/k$. Recall that a k-variety is a geometrically integral, separated, finite-type k-scheme. I learned this material from [Mum08] or [Cai04].

Basic definitions.

Definition 1.1. A *k-group scheme* is a group object G in k-schemes; i.e. a k-scheme equipped with maps $m\colon G \times G \to G$, $i\colon G \to G$, $e\colon \operatorname{Spec} k \to G$ making the expected diagrams commute. A k-group scheme which is a variety is a *group variety*. An *abelian variety* is a proper (smooth) group variety.

In general, we can generalize this to an *abelian scheme* A over a base S, which is a smooth proper group scheme with geometrically connected fibers; i.e. the fibers are abelian varieties of constant dimension. In fact, the smoothness condition is unnecessary over a field; in fact, if X is a geometrically reduced lft k-group scheme, it is automatically k-smooth, because of generic smoothness and stability of $(X_{\overline{k}})^{\mathrm{sm}}$ under $X(\overline{k})$-translations.

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In the case of abelian varieties, the idea is that geometry greatly regulates the group theory of these objects. The main workhorse is the following rigidity lemma:

Lemma 1.2. *Let X be a complete variety (i.e. proper over k), and Y, Z k-varieties. If $f : X \times Y \to Z$ is a morphism such that $f(X \times \{y\}) = \{z\}$ for some $y \in Y(k)$ and $z \in Z(k)$, then f uniquely factors through $\pi_Y : X \times Y \to Y$.*

Proof. See [Cai04, 1.6]. □

As a consequence, we have the following corollaries, justifying the terminology "abelian":

Corollary 1.3. *Every morphism $f : X \to Y$ of abelian varieties over k factors as the composition of a homomorphism and a translation, $t_x : X \to X$ given by $t_x(y) = m(y, x)$.*

Proof. Define $h = t_{i(f(e_X))} \circ f$, so $f = t_{f(e_X)} \circ h$ and $h(e_X) = e_Y$. We claim h is a homomorphism, i.e. the following diagram commutes:

$$\begin{array}{ccc} X \times X & \xrightarrow{h \times h} & Y \times Y \\ {\scriptstyle m_X}\downarrow & & \downarrow{\scriptstyle m_Y} \\ X & \xrightarrow[h]{} & Y \end{array}$$

Denote the top composite by φ and the bottom composite by ψ. Observe that the map

$$\xi : X \times X \xrightarrow{\varphi \times i_Y \circ \psi} Y \times Y \xrightarrow{m_Y} Y$$

has

$$\xi(X \times \{e_X\}) = \xi(\{e_X\} \times X) = \{e_Y\},$$

so by the rigidity lemma, ξ is the constant map with image e_Y, i.e. $\varphi = \psi$. □

Corollary 1.4. *The group structure on an abelian variety X is commutative.*

Proof. Consider the inversion morphism $i : X \to X$. By the previous corollary, it factors as $t_{i(e_X)} \circ h = t_{e_X} \circ h = h$ for a homomorphism h, so i is itself a homomorphism. Thus, the group structure must be commutative. □

Another strength of this theory is the concreteness with which we can deal with line bundles (i.e. invertible sheaves). In general, these results are often stated as the "theorem of the square" and the "theorem of the cube," but I'll just cite a version of these results:

Theorem 1.5. *Let X/k be an abelian variety and $Y \in \mathbf{Sch}/k$. If $f, g, h : Y \to X$ are k-morphisms and $\mathscr{L}$ is an invertible sheaf on X, then*

$$(f+g+h)^*\mathscr{L} \otimes (f+g)^*\mathscr{L}^{-1} \otimes (f+h)^*\mathscr{L}^{-1} \otimes (g+h)^*\mathscr{L}^{-1} \otimes f^*\mathscr{L} \otimes g^*\mathscr{L} \otimes h^*\mathscr{L}$$

is trivial on Y.

Proof. See [Cai04, 1.14]. ◻

Definition 1.6. Let X be a commutative k-group scheme, $n \in \mathbf{Z}$. Let$[n]_X : X \to X$ be the homomorphism corresponding (via Yoneda) to $\times n$ on $X(T)$, for all $T \in \mathbf{Sch}/k$.

Corollary 1.7. *Let X be an abelian variety, $\mathscr{L}$ a line bundle on X. Then, for all $n \in \mathbf{Z}$,*

$$n^*\mathscr{L} \simeq \mathscr{L}^{\otimes \frac{n^2+n}{2}} \otimes (-1)^*\mathscr{L}^{\otimes \frac{n^2-n}{2}}.$$

Proof. Consider $f = n$, $g = 1 = \mathrm{id}$, $h = -1 = i$. Then, the theorem above implies

$$(n+1)^*\mathscr{L} \simeq n^*\mathscr{L}^{\otimes 2} \otimes (n-1)^*\mathscr{L}^{-1} \otimes \mathscr{L} \otimes (-1)^*\mathscr{L},$$

as $[0]_X^*\mathscr{L} = \mathscr{O}_X$ and $1^*\mathscr{L} = \mathscr{L}$. Thus,

$$\begin{aligned}(n+1)^*\mathscr{L} &= \mathscr{L}^{n^2+n} \otimes (-1)^*\mathscr{L}^{n^2-n} \otimes \mathscr{L}^{\frac{n-n^2 2}{\otimes}}(-1)^*\mathscr{L}^{\frac{-n^2+3n-2}{2}} \otimes \mathscr{L} \otimes (-1)^*\mathscr{L} \\ &= \mathscr{L}^{\frac{2n^2+2n-n^2+n+2}{2}} \otimes (-1)^*\mathscr{L}^{\frac{2n^2-2n-n^2+3n-2+2}{2}} \\ &= \mathscr{L}^{\frac{(n+1)(n+2)}{2}} \otimes (-1)^*\mathscr{L}^{\frac{n(n+1)}{2}},\end{aligned}$$

so the theorem follows from induction on n, starting from the trivial cases $n = 0, 1, -1$. ◻

In fact, it turns out that every abelian variety X/k is projective, or equivalently, admits an ample line bundle. We'll return to these ideas when we discuss polarizations.

Isogenies.

Definition 1.8. An *isogeny* $f : X \to Y$ of abelian varieties is a homomorphism satisfying one of the following equivalent conditions:

(a) f is surjective and $\dim X = \dim Y$.

(b) $\ker f$ is a finite group scheme and $\dim X = \dim Y$.

(c) f is finite, flat, and surjective, so $f_*\mathscr{O}_X$ is a locally free $\mathscr{O}_Y$-module of finite rank).

The degree of f is the degree of the field extension $[k(X) : k(Y)]$.

Proposition 1.9. *Let X be an abelian variety, $n \neq 0$. Then, $[n]_X$ is an isogeny of degree $n^{2\dim X}$. Moreover, for char$(k) \nmid n$, $X[n] = \ker[n]_X$ is an étale group scheme of rank $n^{2\dim X}$, and $X[n](k^{sep}) \simeq (\mathbf{Z}/n)^{2\dim X}$.*

Proof. Let $\mathscr{L}$ be an ample line bundle on X (which exists because X is projective). Now, because $[-1]_X$ is an automorphism of X, $(-1)^*\mathscr{L}$ is ample, so for $n \neq 0$, $n^*\mathscr{L}$ is ample. Thus, $\mathscr{L}$ restricted to $X[n]$ is ample and trivial, implying that $\dim X[n] = 0$. To find its degree, observe that if $\mathscr{L}$ is a symmetric line bundle (i.e. $\mathscr{L} = (-1)^*\mathscr{L}$), which always exists by symmetrizing an ample line bundle, we have

$$(\deg n)(\deg \mathscr{L}) = \deg(n^*\mathscr{L}) = \deg\left(\mathscr{L}^{n^2}\right) = n^2 \deg \mathscr{L},$$

whence the claim. The claim about $[n]$ being finite étale follows from computing that it induces multiplication by n on T_0X, which is an isomorphism for $\mathrm{char}(k) \nmid n$. ◻

2. The Dual of an Abelian Variety

Let A/k be an abelian variety. We want to construct a *dual variety* A^t, which should represent the functor of "degree 0 line bundles." Apparently, although I'm uninformed on this point, this generalizes the Weil pairing for elliptic curves, but our primary motivation is that a polarization, which "ties together" A and A^t, is necessary to simplify the moduli of certain kinds of abelian varieties; this will probably be covered in next week's talk. I'll point out [Cai04, §3] as a reference I really like and [Mum08, §13] has a lot of details, but much of what's below comes from [BL07, §4].

The Picard scheme and Dual Variety. Recall that if X is a scheme, the Picard group $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles under tensor product, made into a contravariant functor via pullback. If X is an S-scheme, we might try and define a functor $\mathrm{Pic}(X \times_S -)$ on S-schemes parametrizing line bundles relatively; however, this functor isn't even a Zariski sheaf. Thus, we'll need to be a bit more clever:

Let A/k be an abelian variety, and for any k-scheme T, let $A_T = A \times T$, which admits a map $\epsilon_T\colon T \to A_T$ coming from the identity section of A.

Definition 2.1. A *rigidified sheaf* on A_T is a pair $(\mathscr{L}, \alpha)$ of a line bundle on A_T and a trivialization $\alpha\colon \mathscr{O}_T \to \epsilon_T^* \mathscr{L}$. Then, define the functor $P_{A/k}\colon (\mathbf{Sch}/k)^{\mathrm{op}} \to \mathbf{Set}$ via

$$P_{A/k}(T) = \{\text{isomorphism classes of rigidified sheaves } (\mathscr{L}, \alpha) \text{ on } A_T\},$$

where $T' \to T$ induces $P_{A/k}(T) \to P_{A/k}(T')$ via pullback.

Note that a rigidified sheaf has no nontrivial automorphisms, as if $h\colon \mathscr{L} \to \mathscr{L}$ is an automorphism, it is an element of $\Gamma(A_T, \mathscr{O}_{A_T}) = \Gamma(T, \mathscr{O}_T)$ with $e^*h = 1$, so $h = \mathrm{id}$. Moreover, there is an obvious group structure of $P_{A/k}(T)$ under tensor product of line bundles and their rigidifications. Thus, we expect the following:

Theorem 2.2. *The functor $P_{A/k}$ is represented by a group scheme $\mathrm{Pic}_{A/k}$, with a universal rigidified sheaf $(\mathcal{P}, \nu)$ on $A \times \mathrm{Pic}_{A/k}$; for any $(\mathscr{L}, \alpha)$ on A_T, there exists $g\colon T \to \mathrm{Pic}_{A/k}$ with $(\mathrm{id} \times g)^*(\mathcal{P}, \nu) \simeq (\mathscr{L}, \alpha)$.*

Definition 2.3. The *dual abelian variety* of A is the connected component of the identity, $A^t := \mathrm{Pic}^0_{A/k}$. The restriction of $(\mathcal{P}, \nu)$ to A^t is the *Poincaré sheaf*, $(\mathcal{P}^0, \nu^0)$.

Thus, A^t has a natural group structure, and facts from the theory of algebraic groups implies that A^t is geometrically connected, finite type, and proper. In addition, $\mathrm{Pic}_{A/k}$ is smooth, and A^t is thus an abelian variety with $\dim A^t = \dim A$.

Consider a map $f\colon X \to Y$. Then, we can consider the pullback by $f \times 1\colon X \times Y^t \to Y \times Y^t$ of $(\mathcal{P}^0, \nu^0)_Y$, giving an $(e_X \times \mathrm{id}_{Y^t})$-rigidified line bundle $(f \times 1)^*(\mathcal{P}^0, \nu^0)_Y$ on $X \times Y^t$. This gives a unique map $f^t\colon Y^t \to X^t$ satisfying

$$(\mathrm{id}_X \times f^t)^*(\mathcal{P}^0, \nu^0)_X = (f \times \mathrm{id}_{Y^t})^*(\mathcal{P}^0, \nu^0)_Y.$$

Definition 2.4. $f^t\colon Y^t \to X^t$ is a morphism of abelian varieties, called the *dual morphism.*

Theorem 2.5. *If $\mathcal{P}^0$ is the Poincaré sheaf on $A \times A^t$, viewed as a bundle on $A^t \times A$, it corresponds to a morphism $A \to (A^t)^t$. This double duality morphism is an isomorphism.*

Polarizations and the Rosati Involution.

Definition 2.6. A *polarization* of A is a symmetric morphism $\varphi\colon A \to A^t$ such that

$$\Delta_A^*(1 \times \varphi)^* \mathcal{P}_A = (1, \varphi)^* \mathcal{P}_A$$

is ample on A. In particular, φ is an isogeny.

Here, a symmetric morphism is $\varphi\colon A \to A^t$ such that the dual $\varphi^t\colon A \simeq (A^t)^t \to A^t$ is the same as φ, or equivalently, the L_φ is isomorphic to the pullback of L_φ by the interchange map on $A \times_k A$. Thus, polarization is a kind of positivity condition; we can imagine that symmetric morphisms are like symmetric bilinear forms, while a polarization is like a positive definite form [Con].

Definition 2.7. A *principal polarization* of a A is a polarization $\varphi\colon A \to A^t$ which is an isomorphism. A *principally polarized abelian variety* is a pair (A, φ) of an abelian variety A and a principal polarization $\varphi\colon A \simeq A^t$.

Definition 2.8. For any polarization $\varphi\colon A \to A^t$, the *Rosati involution* † on $\mathrm{End}_k^0(A)$ is

$$\lambda \mapsto \lambda^\dagger := \varphi^{-1} \circ \lambda^t \circ \varphi.$$

This is a $\mathbf{Q}$-algebra anti-automorphism.

These structures, while not important for Serre-Tate, are key parts of the story of abelian varieties, and they'll come up next week.

3. p-Divisible Groups and Serre-Tate

The Serre-Tate theorem tells us that the deformation theory of abelian varieties is controlled by their p-divisible groups, where p-divisible groups themselves are rather rigid kinds of "ind-finite flat group schemes." One helpful perspective to take is that the deformation theory of some objects (bundles, groups, elliptic curves, etc.) is studying the (formal) smoothness of the moduli space of said objects. Thus, we'll show how to associate a p-divisible group to an abelian variety, and it'll turn out that any deformation of said p-divisible group will "fix" a deformation of the whole variety. This is the key nonformal input to Lurie's theorem on lifting sheaves of E_∞-rings, and has a wide variety of applications in number theory and other subjects. The material on p-divisible groups comes from [EvdGM] and [You15], while the proof of Serre-Tate presented is due to Drinfeld, and can be found in section 1 of [Kat81].

p-Divisible Groups.

Definition 3.1. A p-divisible group of height h G over some base scheme S is an fppf abelian sheaf such that

(a) G is p-divisible, i.e. $[p]\colon G \to G$ is an fppf-epimorphism,

(b) G is p-power torsion, i.e. $G = \operatorname{colim} G[p^n]$,

(c) $G[p]$ is a finite flat group scheme of order p^h.

This definition is due to Grothendieck [You15], but perhaps a more familar one, due to Tate, is to considera ind-system G_n, $n \geq 1$ of group schemes over S, where each G_n is finite flat of order p^{nh}, and for all $n \geq 1$, the sequence

$$1 \to G_n \to G_{n+1} \xrightarrow{p^n} G_{n+1}$$

is exact; i.e. G_n should be $G[p^n]$ for $G = \operatorname{colim} G_n$. Note that being a p-divisible group is more than being literally p-divisible, so sometimes these are called Barsotti-Tate groups. Let $\mathbf{BT}_n^p(-)$ denote the functor sending S to the category of p-divisible groups over S.

Definition 3.2. Let A/k be an abelian variety, p a prime number. Then, the *p-divisible group of A* is the p-divisible group given by

$$X[p^\infty] = \operatorname{colim}\left(X[p] \to X[p^2] \to X[p^3] \to\right)$$

where the maps are the natural inclusions. Note that $X[p^\infty]$ has height $2\dim(X)$.

This construction is also functorial: if $f\colon X \to Y$ is a homomorphism, we get a homomorphism $f[p^\infty]$ of p-divisible groups.

Proof of Serre-Tate.

Theorem 3.3 (Serre-Tate)**.** *Let R be a ring in which p is nilpotent, $I \subset R$ is a nilpotent ideal, $R_0 \coloneqq R/I$. Let $\mathbf{Ab}(R)$ be the category of abelian schemes over R, and let $\mathbf{Def}(R, R_0)$ be the category of triples*

$$(A_0, G, \epsilon)$$

with A_0 an abelian scheme over R_0, a p-divisible group G over R, and an isomorphism $\epsilon\colon G_0 \simeq A_0[p^\infty]$. Then, the functor

$$\begin{aligned}\Phi\colon \mathbf{Ab}(R) &\to \mathbf{Def}(R, R_0)\\ A &\mapsto (A_0, A[p^\infty], \textit{natural } \epsilon)\end{aligned}$$

is an equivalence of categories.

More geometrically, the picture is that we have a scheme S_0 (= Spec R_0), an infinitesimal thickening S, with p locally nilpotent on S. Then, the category of abelian varieties on S is equivalent to the category of abelian varieties on S_0 equipped with a lift of its p-divisible group. It's a really deep theorem due to Messing that every p-divisible group on such an S is formally smooth. The following proof is due to Drinfeld, as reproduced in [Kat81, §1], although I owe much of my intuition to the fantastic blog post [You15]. To start, we'll need a workhorse lemma from the realm of formal geometry.

Lemma 3.4. *Let $N \geq 1$ be an integer which kills R, and I a nilpotent ideal, with $I^{\nu+1} = 0$. Let G, H be fppf abelian sheaves on R (i.e. on $\mathbf{CAlg}_R$), such that*

(a) G is N-divisible,

(b) $\widehat{H}$ is locally representable by a formal Lie group

(c) H is formally smooth.

Let G_0/H_0 denote the fibers of G, H on R_0. Then

(1) The groups $\mathrm{Hom}_{\mathbf{Grp}(R)}(G, H)$ and $\mathrm{Hom}_{\mathbf{Grp}(R_0)}(G_0, H_0)$ are N-torsionfree,

(2) The "reduction mod I" map $\mathrm{Hom}(G, H) \to \mathrm{Hom}(G_0, H_0)$ is injective,

(3) For any $f_0 \colon G_0 \to H_0$, there exists a unique homomorphism "$N^\nu f$" lifting $N^\nu f_0$.

(4) A homomorphism $f_0 \colon G_0 \to H_0$ lifts to $f \colon G \to H$ iff "$N^\nu f$" annihilates the subgroup $G[N^\nu] = \ker([N^\nu]_G)$.

Proof. See [Kat81, 1.1.3]. (1) and (2) follow from the fact that G, thus G_0 are N-divisible, and $H_I(A) = \ker(H(A) \to H(A/IA))$ is killed by N^ν.[1] We can write down a lift "$N^\nu f$" as, for any $A \in \mathbf{CAlg}_R$,

$$\begin{array}{ccc} G(A) & \xrightarrow{\text{“}N^\nu f\text{”}} & H(A) \\ \downarrow & & \uparrow{\scriptstyle N^\nu \times \text{any lift}} \\ G(A/IA) & \xrightarrow[f_0]{} & H(A/IA) \end{array}$$

The right vertical morphism exists because we can lift $H(A/IA) \to H(A)$ via formal smoothness, and is well-defined because the indeterminacy, in $H_I(A)$, is killed by N^ν.

Finally, for (4), in one direction, note that any lift f must satisfy $N^\nu f =$ "$N^\nu f$", as both lift $N^\nu f_0$, so "$N^\nu f$" will annihilate $G[N^\nu]$. Conversely, if "$N^\nu f$" annihilates $G[N^\nu]$, then N-divisibility gives the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G[N^\nu] & \longrightarrow & G & \xrightarrow{N^\nu} & G & \longrightarrow & 0 \\ & & & & {\scriptstyle \text{“}N^\nu f\text{”}}\downarrow & \swarrow{\scriptstyle \exists f} & & & \\ & & & & H & & & & \end{array}$$

Now, to see that the homomorphism $f \colon G \to H$ lifts f_0, note that the reduction f/I satisfies $N^\nu f/I = N^\nu f_0$. Thus, as $\mathrm{Hom}(G_0, H_0)$ has no N-torsion, we conclude $f/I = f_0$, thus completing the proof. □

[1] *These* facts, in turn, follow from the basic yoga of formal Lie groups, i.e. formal groups which are formal schemes over Spf A locally of the form Spf $A[[T_1, \dots, T_m]]$; these objects are what those of us coming from homotopy theory probably think of when we hear "formal group." The proof of H_I being killed by N^ν boils down to a computation of the $[N]$-series with coordinates on $\widehat{H}$.

Now, let $N = p^n$, for p a prime. We can show Φ is an equivalence in two steps:

Lemma 3.5. $\Phi\colon \mathbf{Ab}(R) \to \mathbf{Def}(R, R_0)$ *is fully faithful.*

Proof. Let $A, B \in \mathbf{Ab}(R)$ be abelian schemes over R. Say we are given a homomorphism $f[p^\infty]\colon A[p^\infty] \to B[p^\infty]$ of p-divisible groups over R and a homomorphism $f_0\colon A_0 \to B_0$ such that $f_0[p^\infty] = (f[p^\infty]_0)$. We want to construct a unique homomorphism $f\colon A \to B$ inducing both $f[p^\infty]$ and f_0.

Using the above lemma, which applies to , uniqueness is automatic from injectivity of $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A_0, B_0)$. Now, consider the lift "$N^\nu f$" of $N^\nu f_0$. We know that the map "$N^\nu f$"$[p^\infty]$ must lift $N^\nu(f_0[p^\infty])$, and as $N^\nu f[p^\infty]$ is already such a lift, we know "$N^\nu f$"$[p^\infty] = N^\nu(f[p^\infty])$, so "$N^\nu f$" kills $A[N^\nu]$. Thus, the lemma gives a lift F of f_0, with $F[p^\infty]$ lifting $f_0[p^\infty]$, so that $F[p^\infty] = f[p^\infty]$. □

Lemma 3.6. Φ *is essentially surjective.*

Proof. Say we are given $(A_0, G, \epsilon) \in \mathbf{Def}(R, R_0)$. Our goal is to lift this data to an abelian scheme A over R which maps to this triple. First, it turns out that we can always find an abelian scheme B over R which lifts A_0; said more intuitively, the "moduli of abelian schemes is smooth," at least in spirit – one can check by hand that the obstruction class to this lifting vanishes by using the multiplication on A_0. Let $\alpha_0\colon B_0 \xrightarrow{\simeq} A_0$, which induces an isomorphism of p-divisible groups $\alpha_0[p^\infty]$. Now $N^\nu \alpha_0[p^\infty]$ uniquely lifts to a morphism of p-divisible groups

$$\text{“}N^\nu \alpha[p^\infty]\text{”}\colon B[p^\infty] \to G,$$

and this turns out to be an isogeny. The canonical lift of $N^\nu \times (\alpha_0[p^\infty])^{-1}$, from G to $B[p^\infty]$ composes with this map in either direction to give the endomorphism $N^{2\nu}$, so we have an inverse up to isogeny.

Therefore, we get a short exact sequence

$$0 \to K \to B[p^\infty] \to G \to 0,$$

with $K \subset B[N^{2\nu}]$. By algebraic geometry magic ("applying the criterion of flatness fiber-by-fiber") we can conclude that "$N^\nu \alpha[p^\infty]$" is flat, as the reduction mod I, multiplication by N^ν times an isomorphism, is itself flat. Thus, K is a finite flat subgroup of $B[p^{2n\nu}]$, so we can define the quotient abelian scheme $A := B/K$. Now, K lifts $B_0[N^\nu]$, so A lifts $B_0/B_0[N^\nu] \simeq B_0 \simeq A_0$, and we get a compatible isomorphism

$$A[p^\infty] \simeq B[p^\infty]/K \xrightarrow{\simeq} G.$$

□

In the context of Lurie's theorem, we can restate this as follows [Dav22, 2.7]. Suppose $\widetilde{R} \to R$ is a square-zero extension of discrete commutative rings, with p nilpotent. Serre-Tate says that the diagram of 1-groupoids

$$\begin{array}{ccc} \mathbf{Ab}_g(\widetilde{R})^{\simeq} & \longrightarrow & \mathbf{Ab}_g(R)^{\simeq} \\ \downarrow & & \downarrow \\ \mathbf{BT}^p_{2g}(\widetilde{R})^{\simeq} & \longrightarrow & \mathbf{BT}^p_{2g}(R)^{\simeq} \end{array}$$

is Cartesian. Thus, the morphism $[p^\infty]\colon \mathscr{M}_{\mathbf{Ab}_g} \to \mathscr{M}_{\mathbf{BT}^p_{2g}}$ of classical moduli stacks sending an abelian variety of dimension g to its associated p-divisible group is formally étale after base change over Spf $\mathbf{Z}_p$.

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